The Fundamental Theorem of Calculus.

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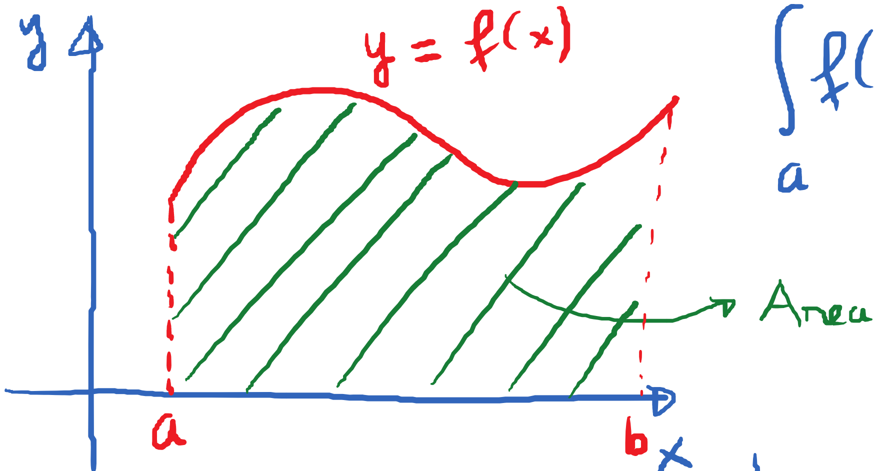
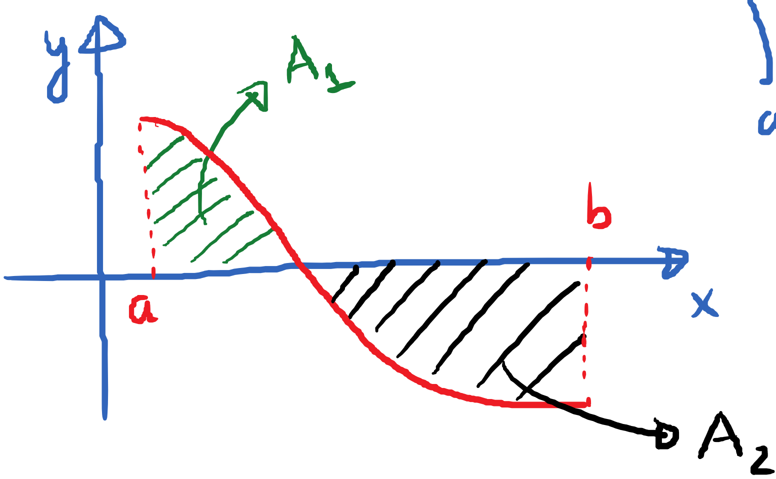


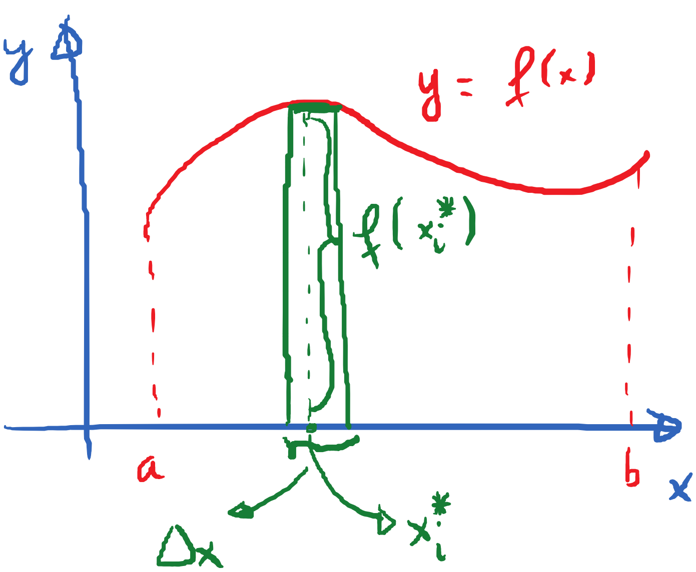
$$\int_a^b f(x)\,dx = \text{area under the graph of } y = f(x),\ a \le x \le b$$



$$\int_a^b f(x)\,dx = A_1 - A_2$$

(signed Area)

Definition of Definite Integral



$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx$$

$$\Delta x = \frac{b-a}{n}$$

The Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$, (i.e. $F'(x) = f(x)$) then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Useful notation: $F(x)\Big|_a^b = F(b) - F(a)$

$$\int_a^b f(x)\,dx = \underbrace{F(x)\Big|_a^b}_{\text{read as } F \text{ evaluated from } a \text{ to } b.} = F(b) - F(a)$$

E.g. ① $\displaystyle\int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{(4)^3}{3} - \frac{(0)^3}{3} = \boxed{\frac{64}{3}}$

② $\displaystyle\int_1^4 \frac{4}{x^2}\,dx = 4\int_1^4 \frac{1}{x^2}\,dx = 4\int_1^4 x^{-2}\,dx = 4\cdot\frac{x^{-1}}{-1}\Big|_1^4 = -\frac{4}{x}\Big|_1^4$

$$= -\frac{4}{4} - \left(-\frac{4}{1}\right) = -1 + 4 = \boxed{3}$$

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Ex.

(3) $\int_{4}^{8}\left(4t^{\frac{5}{2}} - 3t^{\frac{3}{2}}\right)dt$

(4) $\int_{1}^{4}\frac{2-\sqrt{z}}{z^2}\,dz$

(5) $\int_{0}^{\pi/2}\sin(\theta)\,d\theta$

(6) $\int_{\pi/4}^{\pi/2}\csc^2\theta\,d\theta$

(7) $\int_{0}^{5}\sqrt{25-x^2}\,dx$ (Hint: try using the geometric meaning of integral)

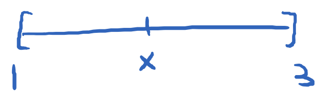
Solved in class.

Average Value of a function f on $[a, b]$.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

FTC, part I

E.g. $\boxed{f(t) = t^2}$, $[1, 3]$



$$\int_1^x f(t)\,dt = \int_1^x t^2\,dt = \frac{t^3}{3}\Big|_1^x = \boxed{\frac{x^3}{3} - \frac{1}{3}} \rightarrow F(x)$$

the answer is a function in x

$$F(x) = \int_1^x f(t)\,dt$$

$$\boxed{F'(x) = x^2} \qquad \frac{d}{dx}\big(F(x)\big) = f(x)$$

FTC part I

Suppose $f(t)$ is a continuous function on $[a, b]$.

Define the function $F(x)$ by the following integral:

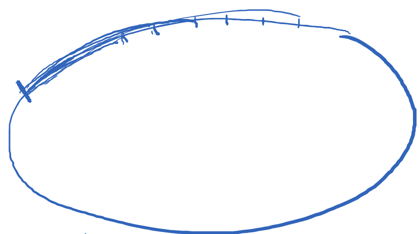
$$F(x) = \int_a^x f(t)\,dt$$

FTC part I says that

$$F'(x) = f(x) \text{ for every } x \text{ in } [a, b]$$

In short,

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$



$$\cancel{\int_0^x \frac{1}{\sqrt{(1-t^2)(1-4t^2)}}\,dt}$$

$$\int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \qquad \int \left[(1-t^2)(1-k^2t^2)\right]^{-1/2} dt$$

$$\underbrace{\frac{d}{dx}\int_0^{x} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}_{F(x)} \longrightarrow \underbrace{\frac{1}{(1-x^2)(1-k^2x^2)}}_{F'(x)}$$

$$\int t^{-1/2}\,dt$$

E.g

$$\frac{d}{dx}\left(\int_2^{x} e^{-t^2}\,dt\right) = e^{-x^2}$$

$$\frac{d}{dx}\left(\int_{2017}^{x} e^{-t^2}\,dt\right) = e^{-x^2}$$

$$\boxed{\int e^{-t^2}\,dt}$$

$$\cancel{\int e^{-t}} \qquad \iint \qquad \iiint$$

E.g. $\frac{d}{dx}\left(\int_1^{x^4} \sec(t)\,dt\right) = \sec(x^4)\cdot(4x^3)$

In general,

$$\boxed{\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = \boxed{f(g(x))}\; g'(x)}$$

Ex.- (1) $\frac{d}{dx}\left(\int^{\sin x} \sqrt{1-t^2}\,dt\right)$ (2) $\frac{d}{dx}\left(\int^{\pi} \frac{dt}{\quad}\right)$

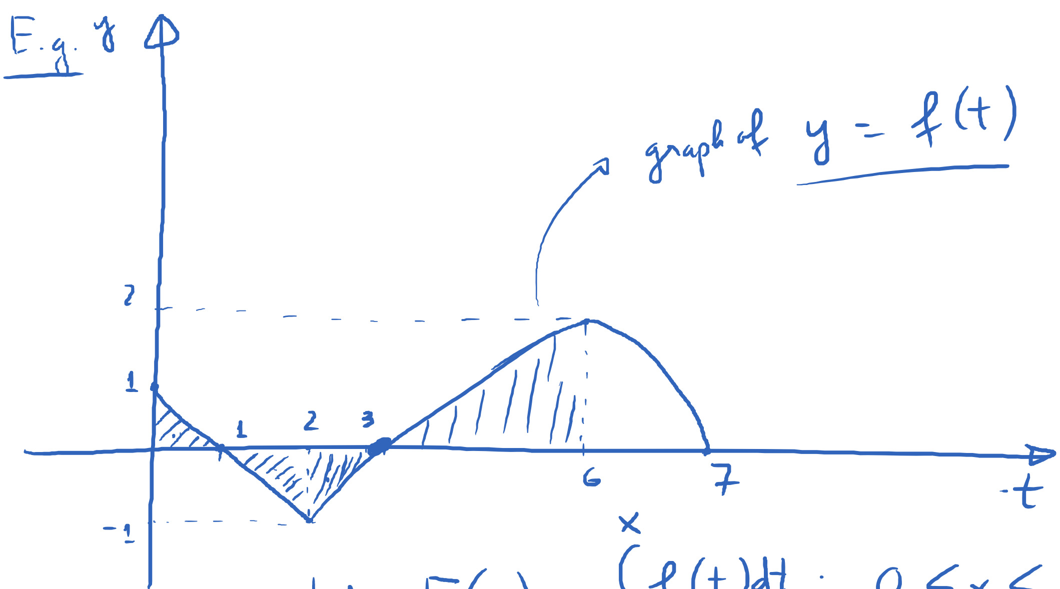
Ex-

(1) $\dfrac{d}{dx}\left(\displaystyle\int_{0}^{\sin x} \sqrt{1-t^2}\, dt\right)$ (2) $\dfrac{d}{dx}\left(\displaystyle\int_{4}^{\pi} \frac{dt}{\sqrt{16-t^2}}\right)$

(3) $\dfrac{d}{dx}\left(\displaystyle\int_{\sqrt{x}}^{2x} \arctan(t)\, dt\right)$

Solved in class.

E.g.



let $F(x) = \displaystyle\int_0^x f(t)dt \; ; \quad 0 \le x \le 7$

(1) Find $\boxed{F(0)}$; $\boxed{F(1)}$; $F(2)$; $F(3)$; $F(6)$

(2) On which interval is F increasing / decreasing.

(3) Find local max / min of F (if any)

→ (1) $F(0) = 0$; $\displaystyle\int_0^0 f(t)dt = 0$

$F(1) = \frac{1}{2}$. $\quad F(1) = \displaystyle\int_0^1 f(t)dt$

$F(2) = \displaystyle\int_0^2 f(t)\,dt = 0 \qquad F(3) = -\frac{1}{2}$; $F(6) = \frac{1}{2}$.

② $F'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$

$F'(x) = f(x) > 0$ on $(0,1) \cup (3,7)$ $\longrightarrow$ F is increasing on $(0,1) \cup (3,7)$

$F'(x) = f(x) < 0$ on $(1,3)$ $\longrightarrow$ F is decreasing on $(1,3)$.

$F'(x) = f(x) = 0$ at $x = 1$; $x = 3$

At $x = 1$ $\longrightarrow$ local max

At $x = 3$ $\longrightarrow$ local min.