Integrals that involve inverse trig functions

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Recall:

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$$

$\vdots$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{1}{1+\frac{x^2}{4}}\cdot\frac{1}{2}$$

$$= \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{1}{2\left(\frac{4+x^2}{4}\right)}$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{1}{\frac{4+x^2}{2}} = \frac{2}{4+x^2}$$

$$\frac{1}{2}\cdot\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{1}{4+x^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) = \frac{1}{4+x^2}$$

$$\frac{d}{dx}\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{9+x^2}$$

$$\frac{d}{dx}\left(\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)\right) = \frac{1}{49+x^2}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right) = \frac{1}{2+x^2}$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\int \frac{1}{16+x^2}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

$$\int \frac{1}{147+x^2}\,dx = \frac{1}{\sqrt{147}}\tan^{-1}\left(\frac{x}{\sqrt{147}}\right) + C$$

In general, $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$\int \frac{dx}{a^2+x^2} \quad \frac{1}{a}$$

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$. $\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \ldots\ldots = \frac{1}{\sqrt{9-x^2}}$

Develop the formula for: $\int \frac{dx}{\sqrt{a^2-x^2}} \stackrel{?}{=} \sin^{-1}\left(\frac{x}{a}\right)$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C}$$

Similarly, there are formulas for integrals that involve $\sec^{-1}(x)$, $\csc^{-1}(x)$, ... in the book.

E.g. (1) $\int \frac{dx}{25+4x^2} = \int \frac{dx}{4\left(\frac{25}{4}+x^2\right)} = \frac{1}{4}\int \frac{dx}{\frac{25}{4}+x^2}$

$= \frac{1}{4} \cdot \frac{1}{\frac{5}{2}} \cdot \tan^{-1}\left(\frac{x}{\frac{5}{2}}\right) + C$

$= \frac{1}{4} \cdot \frac{2}{5} \cdot \tan^{-1}\left(\frac{2x}{5}\right) + C = \boxed{\frac{1}{10}\tan^{-1}\left(\frac{2x}{5}\right) + C}$

(2) $\frac{1}{6}\int \frac{6\sin^{-1}(6t)}{\sqrt{1-36t^2}}\,dt \to du$

Let $u = \sin^{-1}(6t)$. $du = \frac{1}{\sqrt{1-36t^2}} \cdot 6 = \frac{6}{\sqrt{1-36t^2}}\,dt$

$\frac{1}{6}\int u\,du = \frac{1}{6} \cdot \frac{u^2}{2} + C = \frac{u^2}{12} + C = \frac{\left(\sin^{-1}(6t)\right)^2}{12} + C.$

(3) $\int \frac{dx}{x \cdot \sqrt{1-(\ln(x))^2}}$. $\to du$ $\qquad u = \ln x \quad du = \frac{1}{x}\,dx$

$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}(\ln x) + C.$

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}(\ln x) + C.$$

Recap of formulas:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

More examples:

(1)
$$\int \frac{dx}{49+4x^2} = \int \frac{dx}{4\left(\frac{49}{4}+x^2\right)} = \frac{1}{4}\int \frac{dx}{\frac{49}{4}+x^2}$$

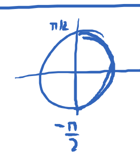
$$= \frac{1}{4}\cdot\frac{1}{\frac{7}{2}}\cdot\tan^{-1}\left(\frac{x}{\frac{7}{2}}\right) + C = \boxed{\frac{1}{14}\cdot\tan^{-1}\left(\frac{2x}{7}\right) + C}$$

(2)
$$-\frac{1}{6}\int \frac{\overbrace{\cos^{-1}(6t)}^{u}\,\overbrace{(-6\,dt)}^{du}}{\sqrt{1-36t^2}}$$

Let $u = \cos^{-1}(6t)$. $du = \dfrac{-1}{\sqrt{1-(6t)^2}}\cdot 6\,dt = \dfrac{-6}{\sqrt{1-36t^2}}\,dt$

$$= -\frac{1}{6}\int u\,du = -\frac{1}{6}\cdot\frac{u^2}{2} + C = -\frac{1}{12}\left(\cos^{-1}(6t)\right)^2 + C.$$

$$* \int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right)$$



$$* \int_{-3}^{x} \frac{1}{\sqrt{9-t^2}}\,dt = \sin^{-1}\left(\frac{t}{3}\right)\Big|_{-3}^{x} = \sin^{-1}\left(\frac{x}{3}\right) - \sin^{-1}(-1)$$

$$= \sin^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}.$$

$$\frac{1}{9}\int \frac{\overbrace{\tan^{-1}(9t)}^{u}\,9\,dt}{1+81t^2} \qquad u = \tan^{-1}(9t)$$

$$du = \frac{1}{\;\;}\,9\,dt = \frac{9}{\;\;}\,dt$$

$$\frac{1}{9}\int \frac{\tan^{-1}(9t)\,9}{1+81t^2}\,dt \qquad u = \tan^{-1}(9t)$$

$$du = \frac{1}{1+(9t)^2}\cdot 9\,dt = \frac{9}{1+81t^2}\,dt$$

$$\frac{1}{9}\int u\,du = \frac{u^2}{18} + C = \frac{(\tan^{-1}(9t))^2}{18} + C$$