4.8. L'Hopital Rule

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L'Hopital Rule for limits of the form $\frac{0}{0}$.

E.g. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} \quad \left(\frac{0}{0}\right)$

L'Hopital Rule: $\lim_{x \to 3} \frac{2x-2}{2x} = \frac{4}{6} = \boxed{\frac{2}{3}}$

L'Hopital Rule for limits of the form $\frac{\infty}{\infty}$:

$$\lim_{x \to \infty} \frac{2x^2 - 5x + 7}{3x^2 + 4} = \frac{\infty}{\infty}$$

L'Hopital Rule: $\lim_{x \to \infty} \frac{4x-5}{6x} \quad \left(\frac{\infty}{\infty}\right)$

L'Hopital Rule: $\lim_{x \to \infty} \frac{4}{6} = \frac{4}{6} = \frac{2}{3}$.

$\frac{0}{0}$; $\frac{\infty}{\infty}$: indeterminate forms

L'Hopital Rule for limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Ex ① $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ ② $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$

③ $\lim_{x \to \infty} \frac{\ln x}{x}$

Solved

Other indeterminate forms

Other indeterminate forms

(Need to manipulate the given function before applying L'Hopital Rule)

$$\begin{cases} 1^{\infty} \longrightarrow \displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} \\ 0^{0} \longrightarrow \displaystyle\lim_{x\to 0} x^{\sqrt{x}} \\ \infty^{0} \longrightarrow \displaystyle\lim_{x\to\infty} \boxed{x}^{\boxed{\frac{1}{x}}} \end{cases}$$

E.g. $$L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$\ln L = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x}$$

$$\ln L = \lim_{x\to\infty} \underbrace{x}_{\infty} \cdot \underbrace{\ln\left(1+\frac{1}{x}\right)}_{0}$$

$$\ln L = \lim_{x\to\infty} \frac{\boxed{\ln\left(1+\frac{1}{x}\right)}}{\boxed{\frac{1}{x}}} \longrightarrow \frac{0}{0}$$

$$\ln L = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

$$e^{\ln L} = e^{1} \longrightarrow \boxed{L = e}$$

$$\ln\left(1+\frac{1}{\ldots}\right) = \lim \frac{\ln\left(1+\frac{1}{x}\right)}{\ldots} = \frac{0}{\ldots} = \ldots$$

$0 \cdot \infty \longrightarrow \lim_{x \to \infty} x \cdot \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \frac{0}{0} \longrightarrow$ L'Hopital

$\infty \cdot 0$

Key: try to use algebra to rewrite the expression to the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ and apply L'Hopital.

$$\infty - \infty \longrightarrow \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

$\infty - \infty$

$$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0}\right) \longrightarrow \text{L'Hopital}$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{1 \cdot \sin x + x \cos x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + 1 \cdot \cos x - x \sin x} = \frac{0}{1 + 1 - 0} = \frac{0}{2} = \boxed{0}$$