4.6. Limits at Infinity.

Thursday, August 3, 2017 10:30 AM

Goals: (1) Find limits at infinity

(2) Find H.A. and V.A. of functions.

Limits at ∞

$y = f(x)$.

Behavior of f when x is very large positively

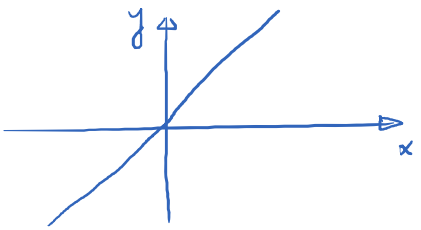
$\longleftrightarrow \lim_{x \to \infty} f(x)$

Behavior of f when x is very large negatively

$\longleftrightarrow \lim_{x \to -\infty} f(x)$

Basic limits at ∞

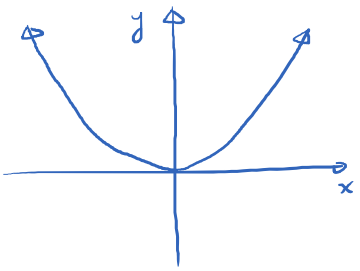
$f(x) = x$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

$f(x) = x^2$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

$f(x) = x^{2017}$ $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = -\infty$

In general, $\lim_{x \to \infty} x^n = \infty$

n is any positive integer

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty \\ -\infty \end{cases}$$

Case 1 : n is even

Case 2 : n is odd

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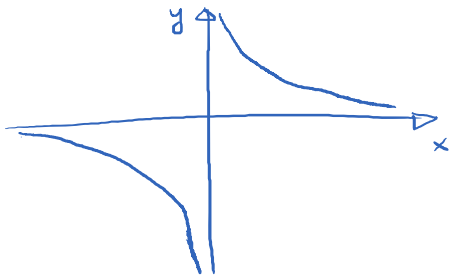
* $f(x) = \frac{1}{x}$.

$\lim_{x \to \infty} f(x) = 0$

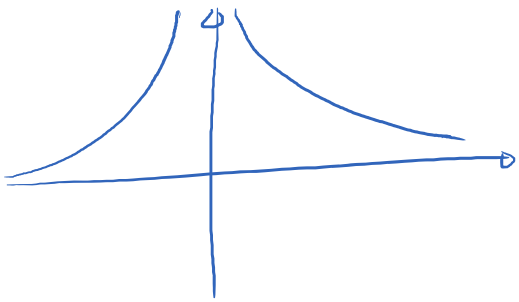
$\lim_{x \to -\infty} f(x) = 0$



* $f(x) = \frac{1}{x^2}$.

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$



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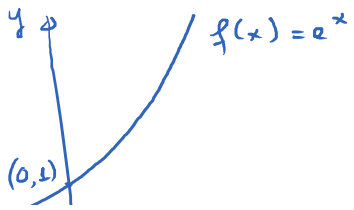
n : positive integer

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

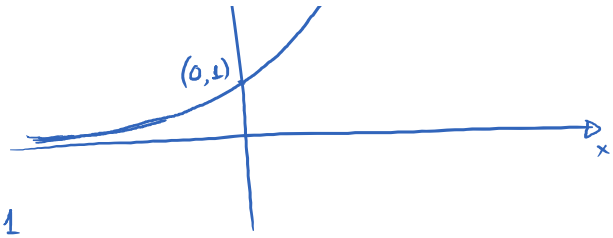
$f(x) = e^x$

$\lim_{x \to \infty} f(x) = \infty$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = 0$

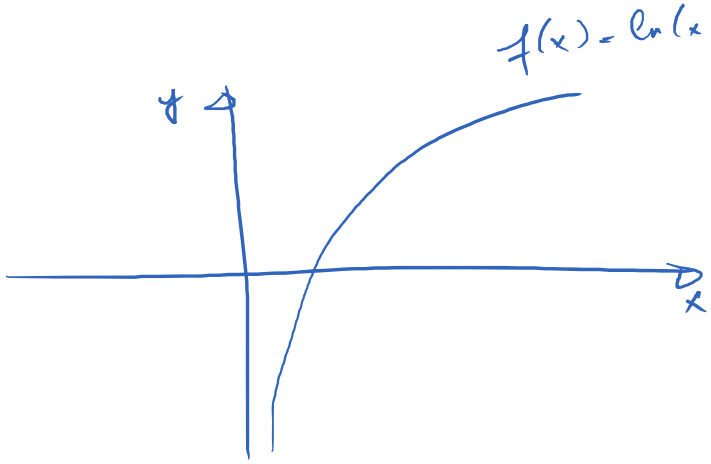


$f(x) = e^{-x} = \frac{1}{e^x}$

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = \infty$

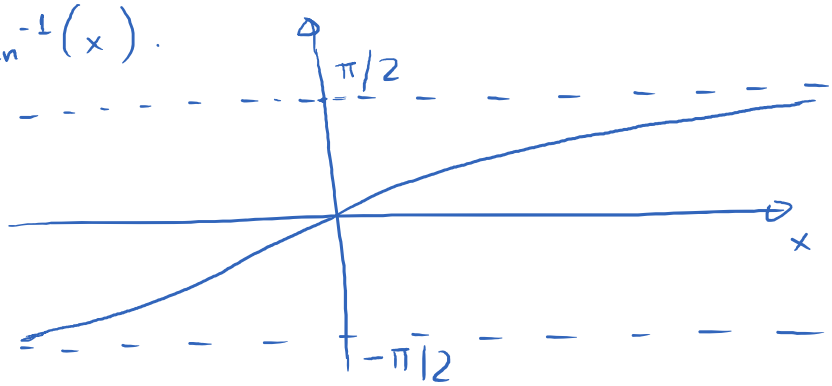
$f(x) = \ln x$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x)$

$f(x) = \tan^{-1}(x)$.



$\lim_{x \to \infty} f(x) = \frac{\pi}{2}$. $\lim_{x \to -\infty} f(x) = -\frac{\pi}{2}$.

E.g. (1) $\lim_{x \to \infty} \left(3 + \frac{2}{x}\right) = 3$

(2) $\lim_{x \to \infty} \frac{x^2 - 2x + 5}{3x^2 + 7}$

1st way: $\lim_{x \to \infty} \frac{(x^2 - 2x + 5)/x^2}{(3x^2 + 7)/x^2} = \lim_{x \to \infty} \frac{1 - \frac{2}{x} + \frac{5}{x^2}}{3 + \frac{7}{x^2}} = \frac{1}{3}$ (the $\frac{2}{x}$, $\frac{5}{x^2}$ and $\frac{7}{x^2}$ terms $\to 0$)

1st way: $\lim_{x\to\infty} \frac{(x^2-2x+5)/x^2}{(3x^2+7)/x^2} = \lim_{x\to\infty} \frac{1-\frac{2}{x}+\frac{5}{x^2}}{3+\frac{7}{x^2}} = \frac{1-0+0}{3} = \frac{1}{3}$

$\lim_{x\to-\infty} f(x) = \frac{1}{3}$.

2nd way: $\frac{x^2-2x+5}{3x^2+7} \approx \frac{x^2}{3x^2} = \frac{1}{3}$

$\lim_{x\to\infty} f(x) = \frac{1}{3}$.

E.x. ① $\lim_{x\to\infty} \frac{x^2-2x+5}{7x+6} = \infty$ ✓ $\approx \frac{x^2}{7x} = \frac{x}{7}$

② $\lim_{x\to-\infty} \frac{x^2-2x+5}{7x+6} = -\infty$ ✓

③ $\lim_{x\to\infty} \frac{x^5-2x+5}{7x+6} = \infty$ $\approx \frac{x^5}{7x} = \frac{x^4}{7}$

④ $\lim_{x\to-\infty} \frac{x^5-2x+5}{7x+6} = -\infty$ ∞ $\frac{x^4}{7}$

⑤ $\lim_{x\to\infty} \frac{49x^2-1}{x^{2017}+10} \approx \frac{49x^2}{x^{2017}} = \frac{49}{x^{2015}} \longrightarrow 0$

In general, Rational function: $\frac{p(x)}{q(x)} = f(x)$

1st case: degree $p(x)$ = degree $q(x)$

$\lim_{x\to\pm\infty} f(x) = \frac{a}{b}$ where a is leading coefficient of top.

b ——— bottom.

2nd case: degree $p(x)$ > degree $q(x)$

$\lim_{x\to\pm\infty} f(x)$ could only be ∞ or $-\infty$

3rd case: degree $p(x)$ < degree $q(x)$

$$\lim_{x \to \pm\infty} f(x) = 0$$

E.g. ① $\lim_{x \to \infty} \dfrac{3x+2}{\sqrt{4x^2+5}}$

$\approx \dfrac{3x}{\sqrt{4x^2}} = \dfrac{3x}{\sqrt{4} \cdot \sqrt{x^2}}$

$= \dfrac{3x}{2 \cdot x} = \boxed{\dfrac{3}{2}}$

② $\lim_{x \to -\infty} \dfrac{3x+2}{\sqrt{4x^2+5}}$

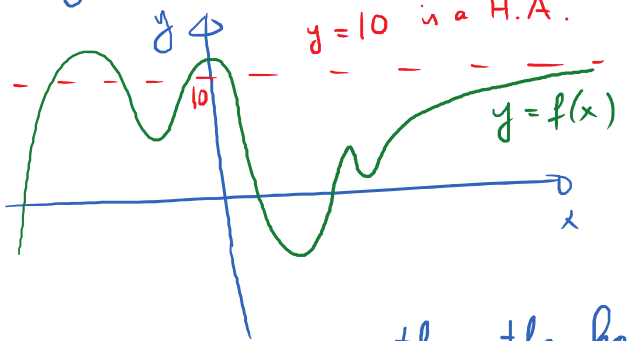
$\approx \dfrac{3x}{\sqrt{4x^2}} = \dfrac{3x}{\sqrt{4} \cdot \sqrt{x^2}}$

$= \dfrac{3x}{2 \cdot (-x)} = -\dfrac{3}{2}$.

E.g. $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x} \approx \dfrac{-e^x}{2e^x} = -\dfrac{1}{2}$

E.g. $\lim_{x \to \infty} \arctan\left(e^x\right) \stackrel{?}{=} \dfrac{\pi}{2}$

Horizontal Asymptotes



$\lim_{x \to \infty} f(x) = 10$

If $\lim_{x \to \infty} f(x) = a$ then the horizontal line $y = a$ is a H.A. of f

If $\lim_{x \to -\infty} f(x) = b$ then ——— $y = b$ is a H.A. of f

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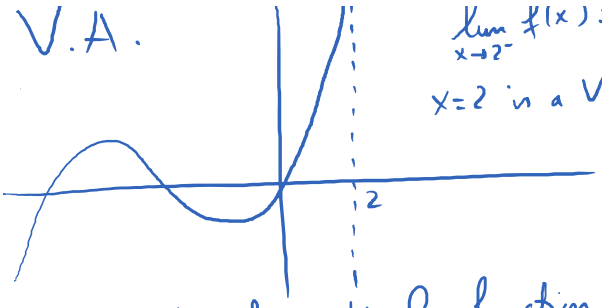
V.A. | |

$\lim_{x \to 2^-} f(x) = \infty$

$x = 2$ is a V.A.

V.A.



$\lim_{x \to 2^-} f(x) = \infty$

$x = 2$ is a V.A.

Find V.A. of rational function $f(x) = \frac{p(x)}{q(x)}$

$f(x) = \frac{x^2 - 9}{x^2 - 4x + 3}$.

H.A. $\boxed{y = 1}$

$f(x) = \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x-1)} = \frac{x+3}{x-1}$

V.A. $\boxed{x = 1}$