4.2. Linear Approximations and Differentials

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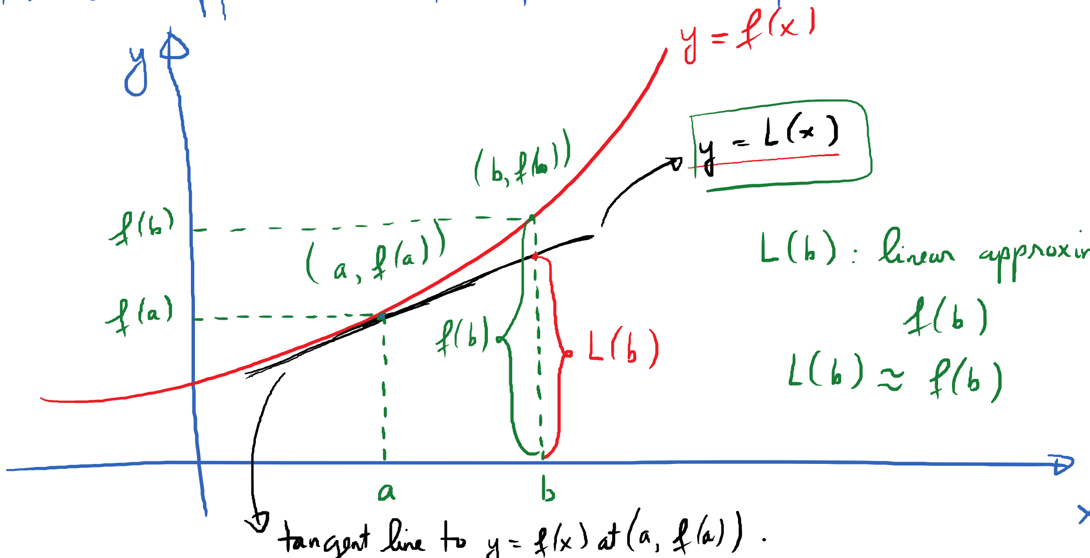
(1) The linear approximation of a function at a point.



$L(b)$: linear approximation to $f(b)$

$L(b) \simeq f(b)$

Equation of tangent line $y = L(x)$

The linear approximation of the function $y = f(x)$ at the point $(a, f(a))$ is the function $\boxed{y = L(x)}$ whose equation is the equation of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

Formula for the linear approximation $L(x)$.

$$\text{Slope} = f'(a)$$

$$\text{Point} = (a, f(a))$$

Point-Slope Equation of tangent line

$$y - f(a) = f'(a)(x - a)$$

$$y = \boxed{f(a) + f'(a)(x-a)}$$

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

$$\boxed{\underbrace{L(x)}_{} = f(a) + f'(a)(x-a)}$$

→ linear approx. for $y = f(x)$ at $(a, f(a))$

E.g. let $f(x) = \sqrt{x}$

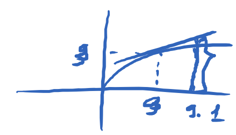
(a) Find the linear approximation to the graph of f at $a = 9$.

(b) Use the linear approximation to estimate $\sqrt{9.1}$.

(a) $L(x) = f(a) + f'(a)(x-a)$

$f(x) = \sqrt{x}$; $a = 9$. $f'(x) = \dfrac{1}{2\sqrt{x}}$

$f(9) = 3$; $f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$



$$\boxed{L(x) = 3 + \frac{1}{6}(x-9)}$$

$$L(x) = 3 + \frac{1}{6}x - \frac{3}{2} \longrightarrow \boxed{L(x) = \frac{x}{6} + \frac{3}{2}}$$

(b) Approximate $\sqrt{9.1}$ by $L(x)$

Plug $x = 9.1$ into $L(x) \longrightarrow$ estimate

$$L(9.1) = 3 + \frac{1}{6}(9.1-9) = 3 + \frac{1}{6}(0.1)$$

$$= 3 + 0.016666$$

$$= 3.01667$$

Ex. (1) $f(x) = e^{x}$

(a) Find linear approximation $L(x)$ when $a = 0$.

(b) Use $L(x)$ to estimate $e^{-0.015}$

(2) $f(x) = \sqrt[3]{x}$. Use linear Approximation to estimate $\sqrt[3]{1001}$.

Solved in class.

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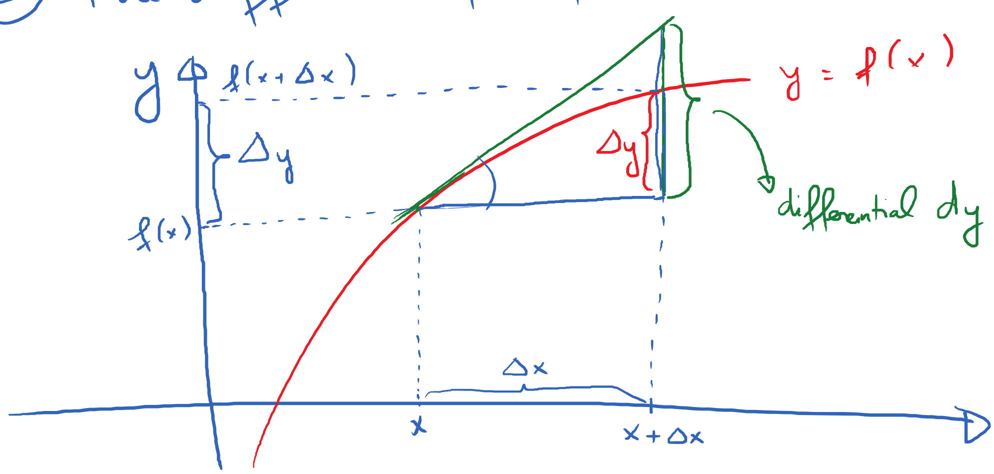
(2) The differential of a function.



Change in y value as we move from x to $x+\Delta x$:

$$\boxed{\Delta y} = f(x+\Delta x) - f(x)$$

let $\boxed{dx = \Delta x}$ $\qquad \dfrac{dy}{dx} = f'(x)$

$$\text{Differential} = \boxed{f'(x)\,dx}$$

⟶ the differential $dy = f'(x)dx$ is an approximation to Δy, which is the actual change in the y values

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Note: $dx = \Delta x$

But $dy \neq \Delta y$.

dy is an approximation for Δy.

Δy is the actual change in y values

$$\Delta y = f(x + \Delta x) - f(x)$$

$$dy = f'(x) \cdot dx$$

E.g. $y = f(x) = x^3 + x^2 - 2x + 1$.

(a) Compare the values of Δy and dy when x change from $\boxed{2}$ to 2.05.

$$dx = \Delta x = 0.05$$

$$\boxed{\Delta y} = f(x + \Delta x) - f(x) = f(2.05) - f(2)$$

$$\approx \boxed{0.7176}$$

$\Delta y \rightarrow$ actual change

$$dy = f'(x) \cdot dx$$

$$f'(x) = 3x^2 + 2x - 2.$$

when $x = 2$. $f'(2) = 3 \cdot (2)^2 + 2 \cdot 2 - 2 = 14$

$f'(2) \cdot (0.05)$

When $x = 2$.

$$dy = f'(2) \cdot (0.05)$$

$$= 14 \cdot (0.05) = \boxed{0.7}$$

approx

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E.g. Suppose the side length of a cube is measured to be 5 cm with an error at most 0.1 cm.

Use this measurement to calculate volume.

Q: Use differential to estimate the error in the computed volume

$x = 5$. $dx \leq 0.1$

$V = x^3$; $dV = 3x^2 dx$

$\leq 3 \cdot (5)^2 \cdot 0.1 = 7.5$

$V = \boxed{125}$ with an error at most $\boxed{7.5}$