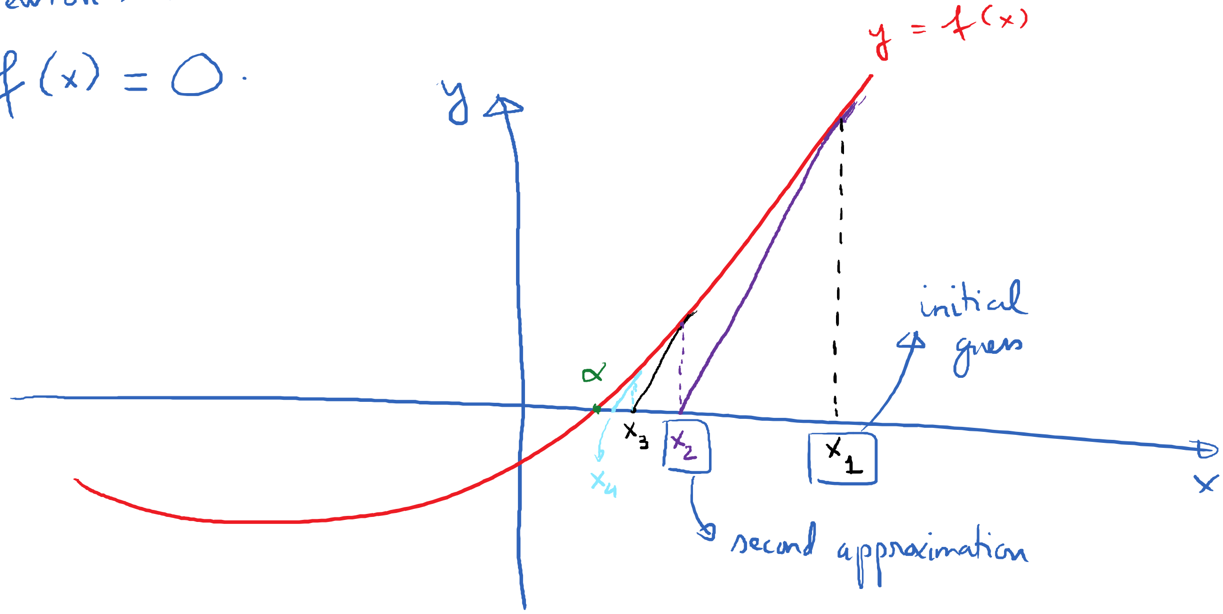
Newton's Method

Tuesday, August 8, 2017 7:29 AM

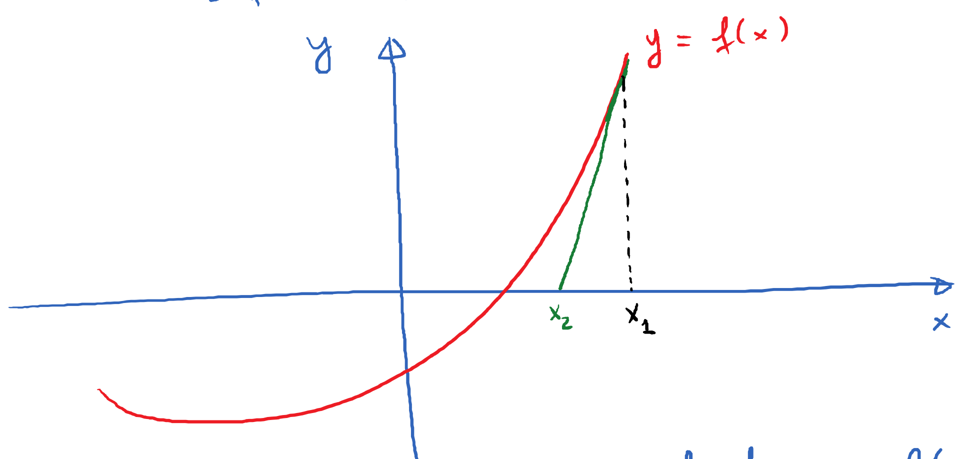
Newton's Method is used to estimate the solution of the equation $f(x) = 0$.



If the function is nice enough, the sequence of approximations $x_1, x_2, x_3, x_4, \dots$ gets closer and closer to the solution $x = \alpha$ of the equation $f(x) = 0$

What is the formula to find the next approximation from the previous one?

i.e. Find x_2 from x_1, x_3 from x_2, x_4 from x_3?



1^{st}: find equation of tangent line to graph of $y = f(x)$ at $x = x_1$.

Slope $f'(x_1)$. Point $(x_1, f(x_1))$

Point-Slope equation: $y - f(x_1) = f'(x_1) \cdot (x - x_1)$

$$\boxed{y = f(x_1) + f'(x_1)(x - x_1)}$$

2^{nd} Find x_2, which is the x-intercept of the tangent line.

Set $y = 0$.

$$f(x_1) + f'(x_1)(x_2 - x_1) = 0$$

$$f'(x_1)\cdot(x_2 - x_1) = -f(x_1)$$

$$x_2 - x_1 = \frac{-f(x_1)}{f'(x_1)}$$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$$\boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}}$$

$$\boxed{x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}}$$

In general

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

For $n = 1, 2, 3, 4, \ldots$ —

E.g. Use Newton's method with the initial approximation $x_1 = 2$ to estimate the solution (root) of the equation $\underbrace{x^3 + 5x - 1}_{f(x).} = 0$.

$$f'(x) = 3x^2 + 5.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{17}{17} = 1 \; ; \; \boxed{x_2 = 1}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1 - \frac{f(1)}{f'(1)} = \overbrace{1 - \frac{5}{8}}^{3/8} \approx \underset{x_3}{0.375}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{3}{8} - \frac{f(3/8)}{f'(3/8)} = \frac{3}{8} - \frac{0.928}{5.422} \approx \underset{x_4}{0.204}$$

E.x. Use Newton's method to approximate $\sqrt{2}$. $\to f(x)$

$\sqrt{2}$ is the solution to the equation $\boxed{x^2 - 2} = 0$

(a) Apply Newton's method with $x_1 = 2$, find x_2, x_3, x_4.

(b) In general, find the formula for x_{n+1} in terms of x_n.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad ; \quad f'(x) = 2x$$

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2}{4} = 1.5 .$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5 - \frac{f(1.5)}{f'(1.5)} \approx 1.417$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.414 .$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + 2}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n} \right)}$$