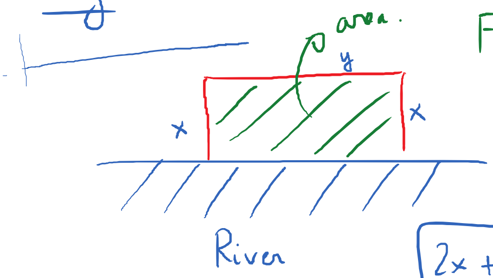
4.7. Optimization Problems

Monday, August 7, 2017 7:39 AM

E.g. Farmer has 2400 ft of fencing.



Find dimensions of rectangle s.t. area is maximized

length = y ; Width = x

$$\text{Area} = xy \leftarrow \text{maximize}$$

$$\boxed{2x + y = 2400} \; ; \quad y = 2400 - 2x$$

$$\text{Area} = A(x) = x \cdot (2400 - 2x) \leftarrow \text{function of 1 variable.}$$

$x \geq 0$; $x \leq 2400$ ft.

Interval for x : $[0, 2400]$

Restate: Maximize of $A(x) = \boxed{x \cdot (2400 - 2x)}$ over $[0, 2400]$

1. Find $A'(x)$ and critical points

$$A(x) = 2400x - 2x^2$$

$$A'(x) = 2400 - 4x = 0 \; ; \quad \boxed{x = 600.} \leftarrow \text{critical point}$$

2. $A(0) = 0$

$$A(600) = 600 \cdot (2400 - 1200) = 600 \cdot 1200 = \boxed{720000}$$

$$A(2400) = 2400(2400 - 4800) < 0$$

When $x = 600$, $A(x)$ is max over the interval $[0, 2400]$

$$2x + y = 2400$$

$$1200 + y = 2400 . \quad y = 1200$$

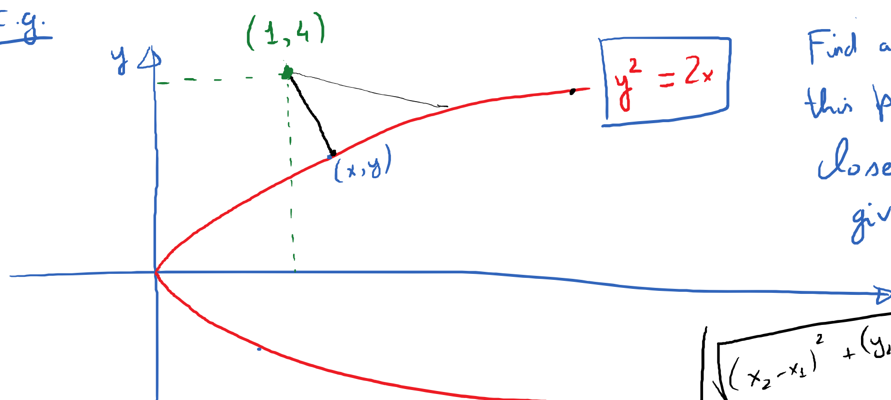
Dimensions to maximize area : $\boxed{x = 600 ; \; y = 1200}$

Dimensions to maximize area : $x = \ldots, \; y = \ldots$

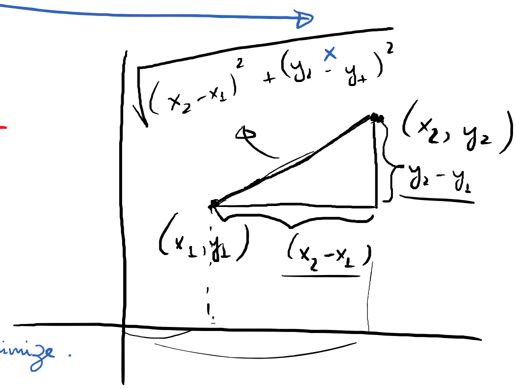
Strategy for solving optimization problems.

1. Understand the problem: given. / want.
2. Draw a diagram
3. Introduce notations: Quantity to maximize or minimize. Q / Unknowns (quantities related to Q)
4. Find an equation that relates Q to the unknown.
5. Turn Q into a function of a single variable.
6. Find the constraint on the variable: closed interval method. / First derivative test.

E.g.



Find a point (x,y) on this parabola that is closest to the given point $(1,4)$



D : distance ← minimize

$$D = \sqrt{(x-1)^2 + (y-4)^2}$$ ← minimize.

$$y^2 = 2x \; ; \quad x = \frac{y^2}{2}$$

$$D(y) = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}$$ ← minimize

y belongs to $(-\infty, \infty)$

Trick: to minimize $\sqrt{}$, we just need to minimize the thing under the $\sqrt{}$

$$d(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 ; \quad y \text{ is in } (-\infty, \infty)$$

Can't use closed interval method $\longrightarrow$ first derivative test.

$$d'(y) = 2\left(\frac{y^2}{2} - 1\right)\cdot y + 2(y-4)$$

$$d'(y) = (y^2-2)\cdot y + 2y - 8 = y^3 - 2y + 2y - 8$$

$$d'(y) = y^3 - 8$$

Find critical points: $y^3 - 8 = 0$; $y = 2$

Make table

y	$-\infty$	1	2	3	∞
$d'(y)$		$-$		$+$	
$d(y)$		$\searrow$	local min	$\nearrow$	

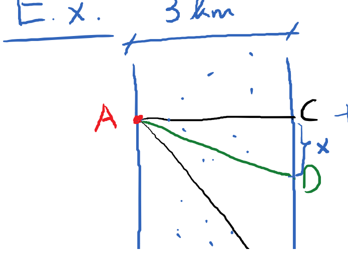
local min $\rightarrow$ absolute min

$d(y)$ is minimized when $y = 2$.

$y^2 = 2x$; $4 = 2x$; $x = 2$.

Point on parabola that is closest to $(1,4)$ is $\boxed{(2,2)}$

Ex.

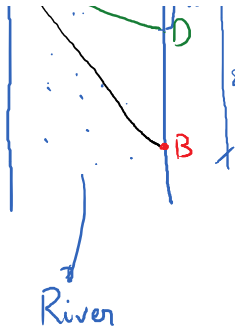


Man launches his boat from A.

Want to reach B.

He could row at a speed of $\boxed{6 \text{ km/h}}$

... at a speed of $\boxed{8 \text{ km/h}}$



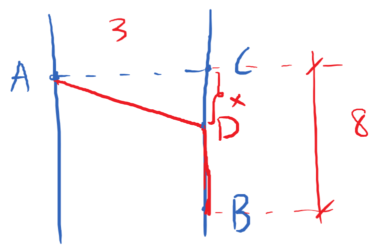
He could row at a speed of $\boxed{6 \text{ km/h}}$

He could run at a speed of $\boxed{8 \text{ km/h}}$

Q: Find the point D on the other side of the river s.t. the time it takes for him to reach B is the shortest.

(Find x s.t. the time to get from A to D to B is the shortest)

Total time to row from A to D and run from D to B



Time it takes to row from A to D:

$$\frac{\text{distance} = AD}{\text{Speed}} = \frac{\sqrt{9+x^2}}{6}.$$

Time it takes to run from D to B:

$$\frac{\text{distance} = BD}{\text{Speed}} = \frac{8-x}{8}$$

Total time:

$$T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8}$$

Find x s.t. $T(x)$ is minimized.

Constraint on x : $[0, 8]$

$\longrightarrow$ closed interval method

$$T'(x) = \frac{1}{6} \cdot \frac{2x}{2\sqrt{9+x^2}} - \frac{1}{8} = \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8}$$

Find critical points:

$$T'(x) = 0 : \quad \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8} = 0$$

$$\frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$

$$\frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$

$$x = \frac{6\sqrt{9+x^2}}{8} = \frac{3\sqrt{9+x^2}}{4}$$

$$x = \frac{3\sqrt{9+x^2}}{4}$$

$$4x = 3\sqrt{9+x^2}$$

$$(4x)^2 = \left(3\sqrt{9+x^2}\right)^2$$

$$16x^2 = 9\cdot(9+x^2)$$

$$16x^2 = 81 + 9x^2$$

$$7x^2 = 81$$

$$x^2 = \frac{81}{7} \; ; \quad x = \sqrt{\frac{81}{7}} = \boxed{\frac{9}{\sqrt{7}}}$$

critical point.

x is in $[0, 8]$

$T(0) = 1.5$ $\qquad T\left(\frac{9}{\sqrt{7}}\right) = \ldots \longrightarrow$ smallest #

$T(8) = \ldots$

$$x = \frac{9}{\sqrt{7}}$$