3.4. Derivatives and Rates of Change.

Thursday, July 20, 2017 7:30 AM

Recall:

Power Rule: $\frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$

Sum / Difference Rule: $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$

Constant Multiple Rule: $\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$

Product Rule: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + g'(x) f(x)$

Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{\left(g(x)\right)^2}$

Thursday, July 20, 2017 7:36 AM

$$\frac{d}{dx}[\sin x] = \cos x. \quad \frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \; ; \quad \frac{d}{dx}[\cot x] = -\csc^2 x$$

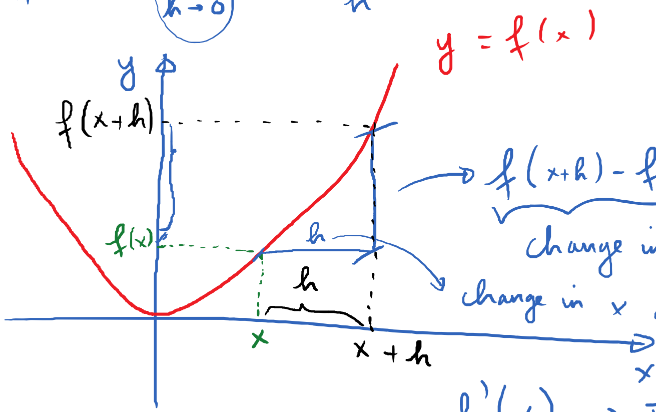
$$\frac{d}{dx}[\sec x] = \sec x \tan x. \quad \frac{d}{dx}[\csc x] = -\csc x \cot x.$$

$f'(x)$ = slope of tangent line to the graph of f at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{Rate of change of } f \text{ on } [x, x+h]} = \frac{\text{change in } y}{\text{change in } x}$$

Average



$\underbrace{f(x+h) - f(x)}$

change in y on $[x, x+h]$

change in x on $[x, x+h]$

$f'(x) \longrightarrow$ Instantaneous R.O.C of f at x

If $f(x)$ is the position function of an moving object, then

$$\frac{\boxed{f(x+h) - f(x)}}{\boxed{h}}$$

$f(x+h) - f(x)$ → change in position = distance traveled on $[x, x+h]$

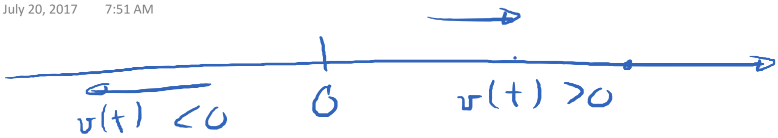
h → time it takes to travel this distance

→ Average Velocity on $[x, x+h]$.

$f'(x)$ = instantaneous velocity at time x

$f''(x)$ = instantaneous acceleration at time x.

$f'''(x)$ = jerk.



Speed: $|v(t)|$

E.g. Position function: $y = 40t - 16t^2$.

* Find average velocity on $[2, 2.1]$

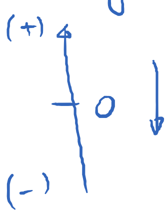
$$\text{Average velocity} = \frac{\text{distance}}{\text{time}} = \frac{y(2.1) - y(2)}{2.1 - 2}$$

$$= \frac{[40\cdot(2.1) - 16(2.1)^2] - [40\cdot 2 - 16\cdot(2)^2]}{(0.1)}$$

* Instantaneous velocity at $t = 2$.

inst. velocity $= \frac{dy}{dx} = 40 - 32t$. at $t = 2$: $40 - 32\cdot 2 = -24\,(m/s)$

Speed at $t = 2$: $|-24| = 24\,m/s$.



* Acceleration at $t = 2$. $\frac{d^2y}{dx^2} = -32\ m/s^2$.

constant acceleration

E.g. $s(t) = 3 - \dfrac{5}{t^2}$; $t > 0$.
(Position)

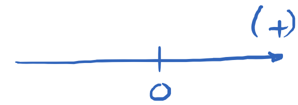
$\underbrace{\dfrac{5}{t^2}}$ $\longrightarrow$ $3 - 5t^{-2}$

Acceleration function $= s''(t)$.

Need: $s'(t) = 10t^{-3}$

So, $s''(t) = -30t^{-4} = \dfrac{-30}{t^4}$

E.g. The position function of a particle moving along an axis is given by the function



$$s(t) = t^3 - 9t^2 + 24t + 4 \;;\; t \geq 0$$

(a) At what time is the particle at rest?

(b) During which time interval is the particle moving from left to right (in (+) direction)? During which time interval is it moving in (−)?

(c) During which time interval is the particle speeding up? Slowing down?

(a) $v(t) = s'(t) = 3t^2 - 18t + 24$

At rest means $v(t) = 0$

$$3t^2 - 18t + 24 = 0$$

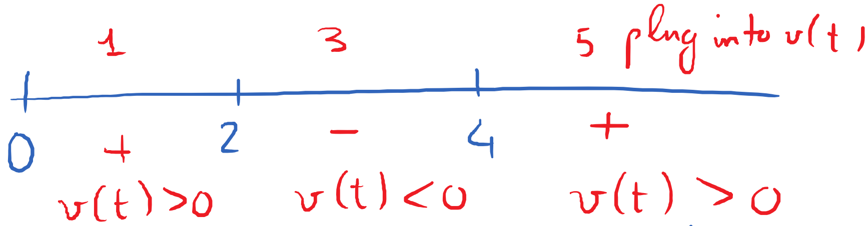
$$3(t^2 - 6t + 8) = 0$$

$$\boxed{3(t-2)(t-4)} = 0$$

$t = 2;\ t = 4.$

So, the particle is at rest at $t = 2$ (s) and $t = 4$ (s).

(b) Find out when $v(t) > 0$; when $v(t) < 0$



Conclusion: Move in (+) direction: $(0,2) \cup (4,\infty)$

Move in (−) direction: $(2,4)$.

e) Speed up? Acceleration and Velocity have the same signs.

$$\begin{cases} v(t) > 0 \\ a(t) > 0 \end{cases} \qquad \begin{cases} v(t) < 0 \\ a(t) < 0 \end{cases}$$

Slow down? Acceleration and velocity have opposite signs.

$$\begin{cases} v(t) > 0 \\ a(t) < 0 \end{cases} \qquad \begin{cases} v(t) < 0 \\ a(t) > 0 \end{cases}$$

$a(t) = 6t - 18$.

So far,

$$v(t) = 3t^2 - 18t + 24 = \underline{3(t-2)(t-4)}$$

$$a(t) = 6t - 18 = 6\underline{(t-3)}$$

t	0		2		3		4		∞
$v(t)$		+	0	−		−	0	+	
$a(t)$		−		−	0	+		+	

zeros for $v(t)$: 2, 4. zero for $a(t)$: 3

Conclusion: speed up on : $(2,3) \cup (4,\infty)$

Slow down : $(0,2) \cup (3,4)$

E.g. $s(t) = \dfrac{t}{4+t^2}$

$$v(t) = s'(t) = \frac{1\cdot(4+t^2) - t\cdot(2t)}{(4+t^2)^2} = \frac{4+t^2-2t^2}{(4+t^2)^2} = \frac{4-t^2}{(4+t^2)^2}$$

$$a(t) = v'(t) = s''(t) = \frac{-2t\cdot(4+t^2)^2 - (4-t^2)\cdot\boxed{\left((4+t^2)^2\right)'}}{(4+t^2)^4}$$

$$\left((4+t^2)^2\right)' = \left((4+t^2)(4+t^2)\right)' = 2t\cdot(4+t^2) + 2t\cdot(4+t^2)$$
$$= 4t\cdot(4+t^2)$$

$$a(t) = \frac{-2t\cdot(4+t^2)^2 - (4-t^2)\cdot 4t\cdot(4+t^2)}{(4+t^2)^4}$$

$$= \frac{-2t\,\cancel{(4+t^2)}\left[4+t^2+(4-t^2)\cdot 2\right]}{(4+t^2)^{\cancel{4}\,3}}$$

$$= \frac{-2t\left[4+t^2+8-2t^2\right]}{(4+t^2)^3} = \frac{-2t\cdot\left[12-t^2\right]}{(4+t^2)^3}$$

$$\boxed{a(t) = \frac{2t\cdot(t^2-12)}{(4+t^2)^3} \quad ; \quad v(t) = \frac{4-t^2}{(4+t^2)^2}}$$

$v(t) = 0$ when? $4 - t^2 = 0$; $t^2 = 4$; $t = \pm 2$.

Since t is time, $\boxed{t=2}$

$a(t) = 0$ when? $2t\cdot(t^2-12) = 0$

$t = 0$; $t = \pm\sqrt{12}$.

$t = 0$; $t = \pm\sqrt{12}$.

Since t is time, $t = 0$; $t = \sqrt{12}$.

t	0		2		$\sqrt{12}$		∞
$v(t)$		$+$	0	$-$		$-$	
$a(t)$	0	$-$		$-$	0	$+$	
		Slow down		Speed up		Slow down.	

$C(x)$: cost function

$C'(x)$: inst. R.O.C for cost = marginal cost.

$P(x)$: profit function

$P'(x)$: inst. R.O.C for profit = marginal profit.

$R(x)$: revenue function.

$R'(x)$: marginal revenue.

$$\boxed{P(x) = R(x) - C(x)}$$