u - Substitution - 2.

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u - Substitution for Definite Integrals

E.g. Evaluate:

$$\frac{1}{6}\int_0^1 \left(\boxed{1+2x^3}\right)^5 \boxed{6x^2\,dx}$$

u

du

$$u = 1 + 2x^3$$

$$du = 6x^2\,dx$$

$$\frac{1}{6}\int u^5\,du$$

1st way of doing the problem → ignore bounds for integration (temporarily)

$$\frac{1}{6}\int u^5\,du = \frac{1}{6}\cdot\frac{u^6}{6} = \frac{u^6}{36} = \frac{(1+2x^3)^6}{36}$$

$$\int_0^1 (1+2x^3)^5\cdot x^2\,dx = \left.\frac{(1+2x^3)^6}{36}\right|_0^1$$

$$= \frac{(1+2\cdot(1)^3)^6}{36} - \frac{(1+2\cdot(0)^3)^6}{36}$$

$$= \frac{3^6}{36} - \frac{1}{36} = \ldots$$

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$2^{\underline{nd}}$ way of doing the problem.

$$\frac{1}{6}\int_0^1 (\underbrace{\boxed{1+2x^3}}_{u})^5 \underbrace{6x^2\,dx}_{du}$$

$u = 1+2x^3$. $\quad du = 6x^2\,dx$

New bounds for u:

$x = 0 \longrightarrow u = 1$

$x = 1 \longrightarrow u = 3$

$\Big\}$ bounds for u are $u = 1$ to $u = 3$

$$\frac{1}{6}\int_1^3 u^5\,du = \frac{1}{6}\,\frac{u^6}{6}\Bigg|_1^3 = \frac{u^6}{36}\Bigg|_1^3$$

$$= \frac{(3)^6}{36} - \frac{(1)^6}{36} = \cdots$$

Integrals that involve Exp. and Log. functions

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Recall some basic derivatives

$$\frac{d}{dx}\left(e^x\right) = e^x \quad ; \quad \frac{d}{du}\left(e^u\right) = e^u$$

$a > 0$, $a \neq 1$, $a \neq e$

$$\frac{d}{dx}\left(a^x\right) = a^x \cdot \ln a \quad ; \quad \frac{d}{du}\left(a^u\right) = a^u \cdot \ln a$$

→ Integral Formulas

$$\boxed{\int e^x\,dx = e^x + C \, . \qquad \int e^u\,du = e^u + C \, .}$$

$a > 0$, $a \neq 1$, $a \neq e$. (base other than e)

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

__Check:__ $\displaystyle \frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}\left(a^x\right) = \frac{1}{\cancel{\ln a}} \cdot a^x \cdot \cancel{\ln a}$

$$= a^x$$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C \quad ; \quad \int a^u\,du = \frac{a^u}{\ln a} + C}$$

E.g. ① $-\int e^{\boxed{-x}} \underbrace{(-dx)}_{du} \overset{?}{=} -\int e^{u}\,du = -e^{u} + C$

$= \boxed{-e^{-x} + C}$

let $u = -x$. $du = -dx$

② $\int 3^{x}\,dx = \dfrac{3^{x}}{\ln 3} + C$

③ $-\int 3^{-x} \underbrace{(-dx)}_{du} = -\int 3^{u}\,du = -\dfrac{3^{u}}{\ln 3} + C$

let $u = -x$. $du = -dx$

$= \boxed{-\dfrac{3^{-x}}{\ln 3} + C}$

④ $\dfrac{1}{8}\int_{0}^{1} 8x \cdot e^{\boxed{4x^2+3}}\,dx$

let $u = 4x^2 + 3$. $du = 8x\,dx$

$\dfrac{1}{8}\int e^{u}\,du = \dfrac{1}{8}\cdot e^{u} = \dfrac{1}{8} e^{4x^2+3}\Big|_{0}^{1}$

$= \dfrac{1}{8}\cdot e^{7} - \dfrac{1}{8} e^{3} = \boxed{\dfrac{1}{8}\left(e^{7} - e^{3}\right)}$

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(5) $\int e^x \sqrt{1+e^x}\, dx$

let $u = 1+e^x$. $du = e^x dx$

$$\int \sqrt{u}\, du = \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{3}(1+e^x)^{3/2} + C}$$

Ex. Evaluate the given integrals

(1) $\int x^3 e^{x^4} dx$

(2) $\int_1^2 e^{1-x} dx$

(3) $\int_1^2 \frac{e^{1/x}}{x^2} dx$

Solved in class

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Integrals that involve log functions

<u>Recall:</u> $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$

$\frac{d}{du}\left(\ln(u)\right) = \frac{1}{u}.$

<u>Recall:</u>

$$\boxed{\begin{gathered}\int \frac{dx}{x} = \ln|x| + C \\ \int \frac{du}{u} = \ln|u| + C\end{gathered}}$$

E.g. 1

$$\frac{1}{2}\int \frac{2(2x^3+3x)}{x^4+3x^2}\,dx$$

du, u

Let $u = x^4 + 3x^2$. $du = 4x^3 + 6x$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^4 + 3x^2| + C.$$

2

$$-\int_0^{\pi/2} \frac{-\sin x}{1+\cos x}\,dx$$

du, u

Let $u = 1 + \cos x$

$du = -\sin x$

New bounds for u

When $x = 0$, $u = 2$

$x = \frac{\pi}{2}$, $u = 1$

$$-\int_2^1 \frac{du}{u} = \int_1^2 \frac{du}{u} = \ln|u|\Big|_1^2 = \ln 2 - \cancel{\ln 1}^{\,0}$$

$$= \boxed{\ln 2}$$

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③ $\int \tan\theta \, d\theta = -\int \frac{\overbrace{-\sin\theta \, d\theta}^{du}}{\underbrace{\cos\theta}_{u}} = -\int \frac{du}{u}$

let $u = \cos\theta$; $du = -\sin\theta \, d\theta$

$$= -\ln|u| + C = -\ln|\cos\theta| + C$$

$$= \ln|(\cos\theta)^{-1}| + C$$

$$= \boxed{\ln|\sec\theta| + C}$$

Useful formula:

$$\boxed{\int \ln x \, dx = x\ln x - x + C}$$

Check: $\frac{d}{dx}(x\ln x - x) = \ln x + x \cdot \frac{1}{x} - 1$

$$= \ln x + 1 - 1 = \ln x \checkmark$$

$$-\int \frac{\ln x}{x^2}\,dx$$

$\frac{1}{u}$

du

$$u = \frac{1}{x}\ ;\ du = -\frac{1}{x^2}$$

If $u = \frac{1}{x}$, then $x = \frac{1}{u}$.

$$-\int \ln\left(\frac{1}{u}\right)du = -\int \ln\left(u^{-1}\right)du = -\int -\ln(u)\,du$$

$$= \int \ln(u)\,du = u\ln u - u + C$$

$$= \frac{1}{x}\ln\frac{1}{x} - \frac{1}{x} + C$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C.$$