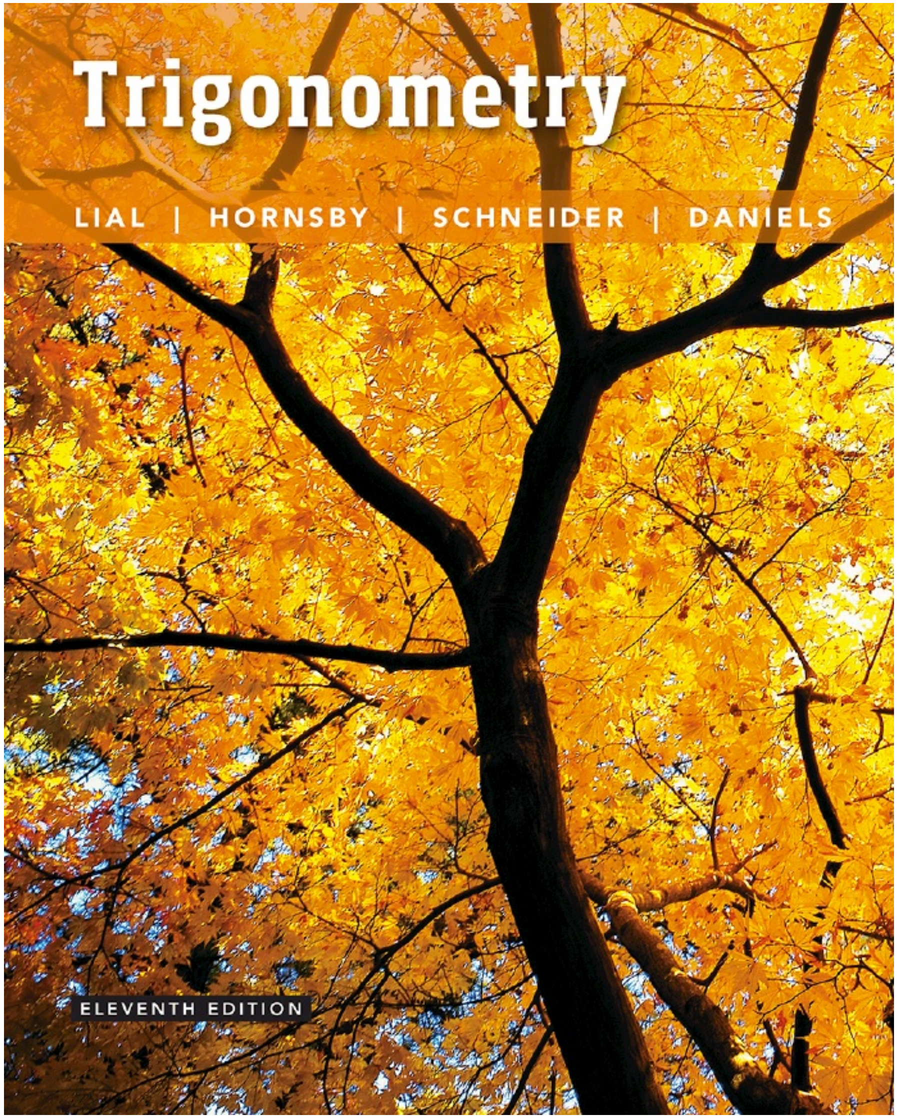
2

Acute Angles and Right Triangles

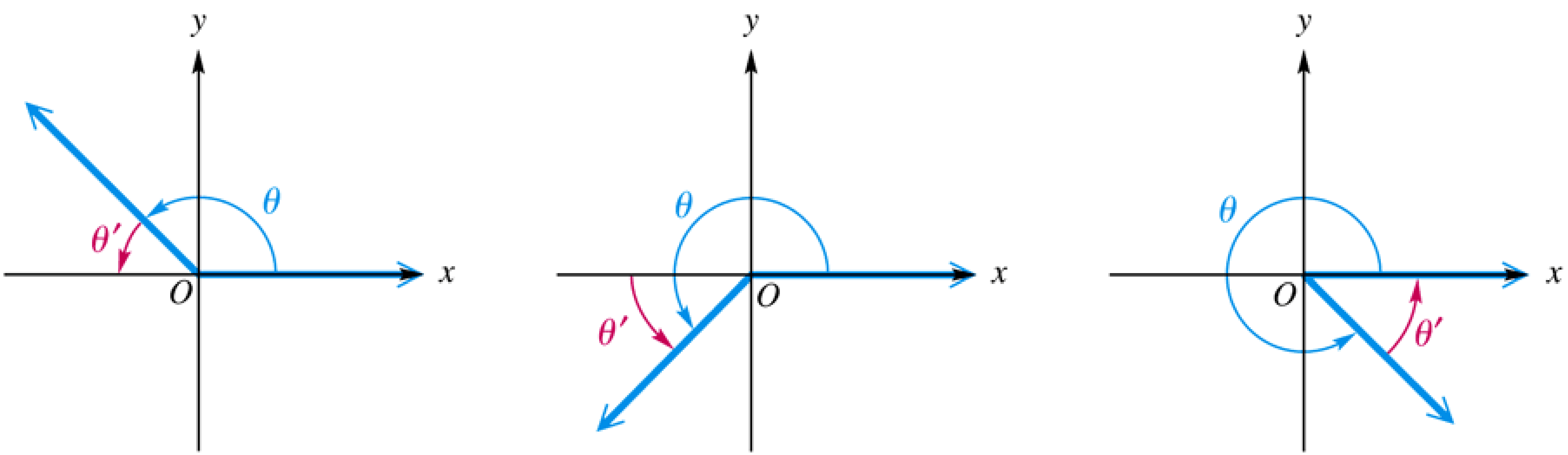


2.2 Trigonometric Functions of Non-Acute Angles

Reference Angles ▪ Special Angles as Reference Angles ▪ Determination of Angle Measures with Special Reference Angles

Reference Angles

A **reference angle** for an angle *θ* is the positive acute angle made by the terminal side of angle *θ* and the *x*-axis.



θ in quadrant II   *θ* in quadrant III   *θ* in quadrant IV

Caution

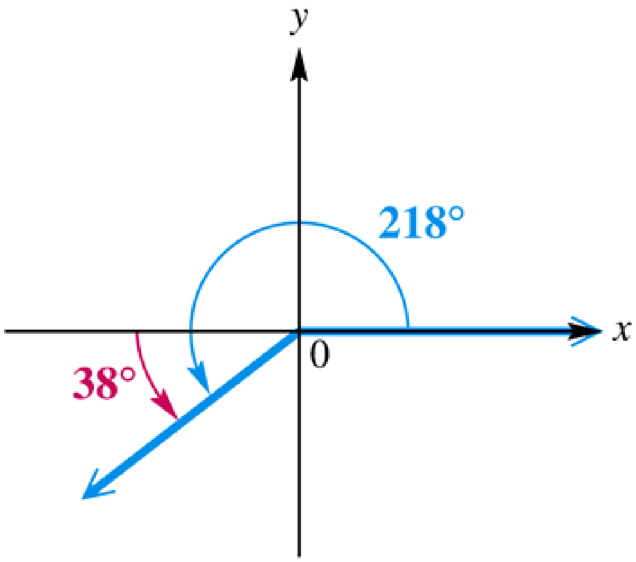
A common error is to find the reference angle by using the terminal side of θ and the *y*-axis.

The reference angle is always found with reference to the x-axis.

 FINDING REFERENCE ANGLES

Find the reference angle for an angle of 218°.

The positive acute angle made by the terminal side of this angle and the *x*-axis is 218° – 180° = 38°.



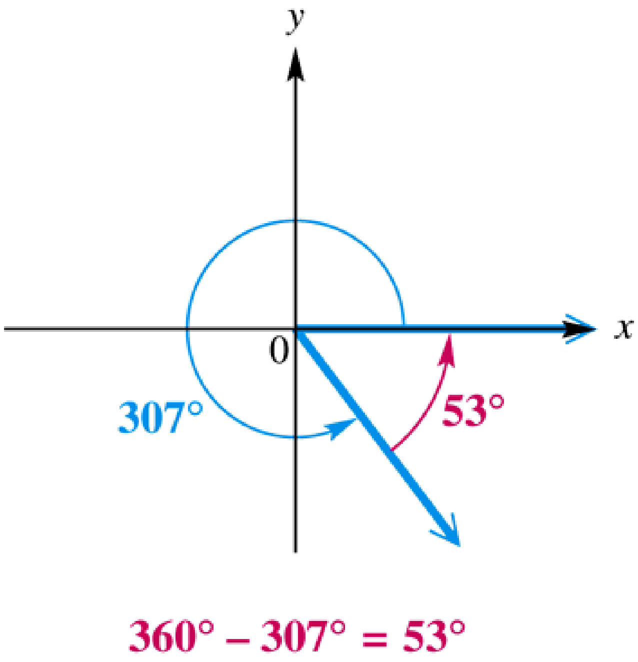
For θ = 218°, the reference angle θ' = 38°.

Example 1(b) FINDING REFERENCE ANGLES

Find the reference angle for an angle of 1387°.

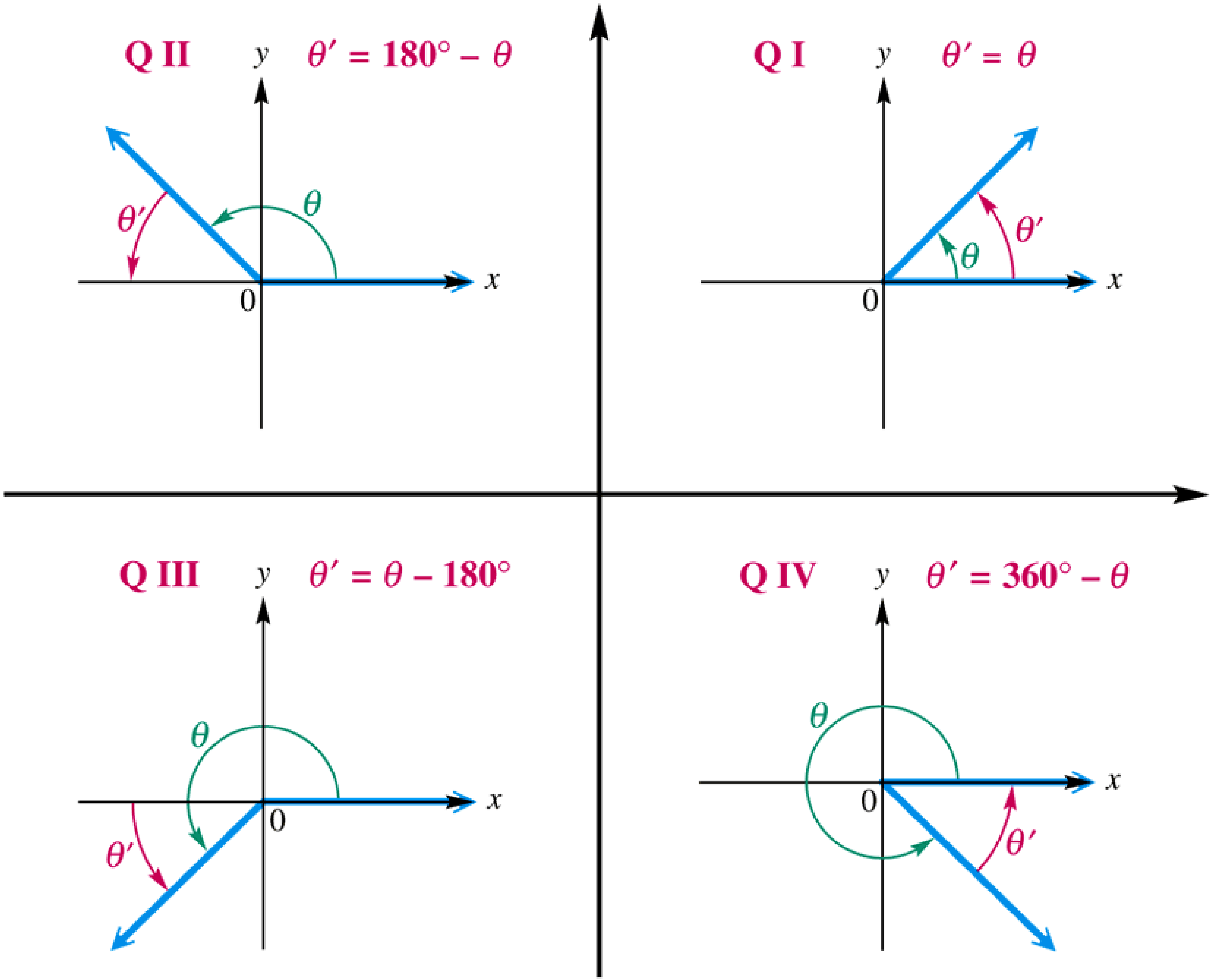
First find a coterminal angle between 0° and 360°.

Divide 1387º by 360 to get a quotient of about 3.9. Begin by subtracting 360° three times.
1387° – 3(360°) = 307°.



The reference angle for 307° (and thus for 1387°) is 360° – 307° = 53°.

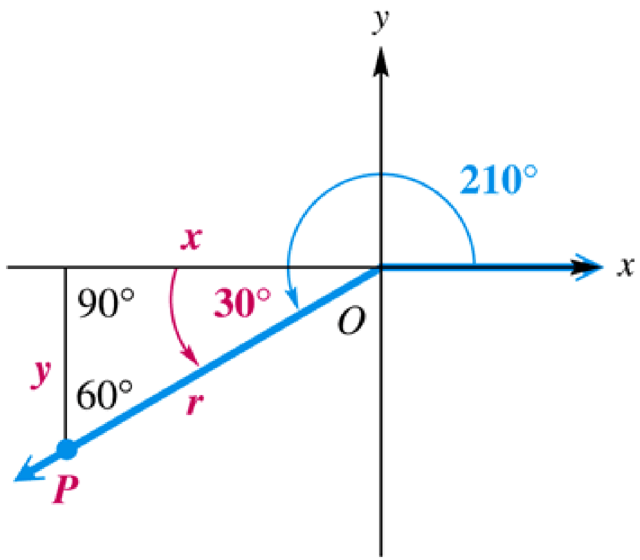
Reference Angle θ' for θ, where $0° < \theta < 360°$



Example 2 FINDING TRIGONOMETRIC FUNCTION VALUES OF A QUADRANT III ANGLE

Find the values of the six trigonometric functions for 210°.

The reference angle for a 210° angle is 210° – 180° = 30°.



Choose point *P* on the terminal side of the angle so the distance from the origin to *P* is 2.

$$r = 2, x = -\sqrt{3}, y = -1$$

Example 2 FINDING TRIGONOMETRIC FUNCTION VALUES OF A QUADRANT III ANGLE (continued)

$$\sin 210^\circ = -\frac{1}{2}$$

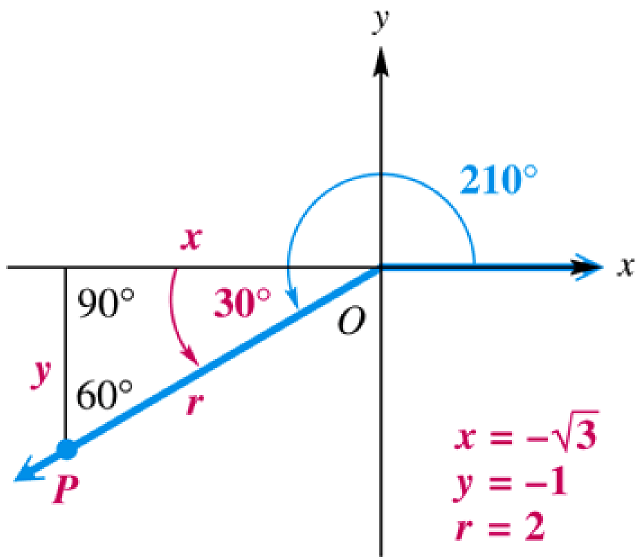
$$\cos 210^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 210^\circ = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 210^\circ = \sqrt{3}$$

$$\sec 210^\circ = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc 210^\circ = -2$$



Finding Trigonometric Function Values For Any Nonquadrantal Angle θ

Step 1 If $\theta > 360°$, or if $\theta < 0°$, find a coterminal angle by adding or subtracting 360° as many times as needed to get an angle greater than 0° but less than 360°.

Step 2 Find the reference angle θ'.

Step 3 Find the trigonometric function values for reference angle θ'.

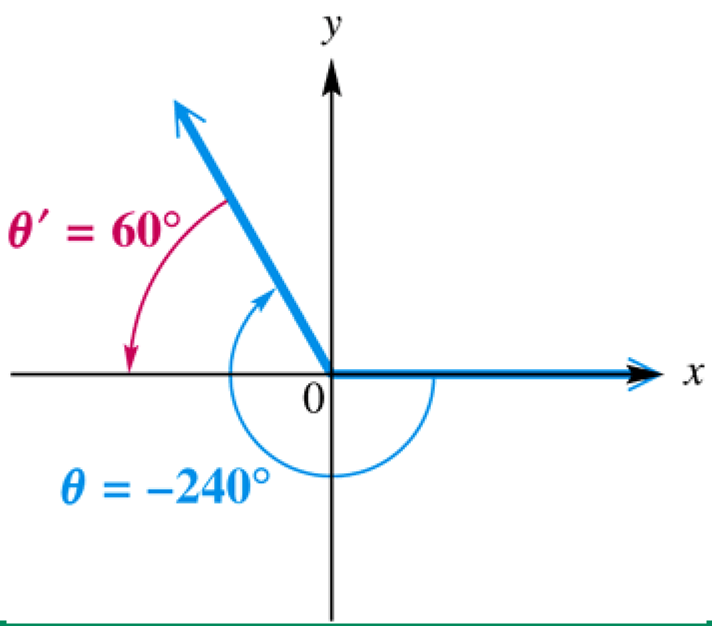
Finding Trigonometric Function Values For Any Nonquadrantal Angle θ (continued)

Step 4 Determine the correct signs for the values found in *Step 3*. This gives the values of the trigonometric functions for angle θ.

Example 3(a) FINDING TRIGONOMETRIC FUNCTION VALUES USING REFERENCE ANGLES

Find the exact value of cos(–240°).

Since an angle of –240° is coterminal with an angle of –240° + 360° = 120°, the reference angle is 180° – 120° = 60°.



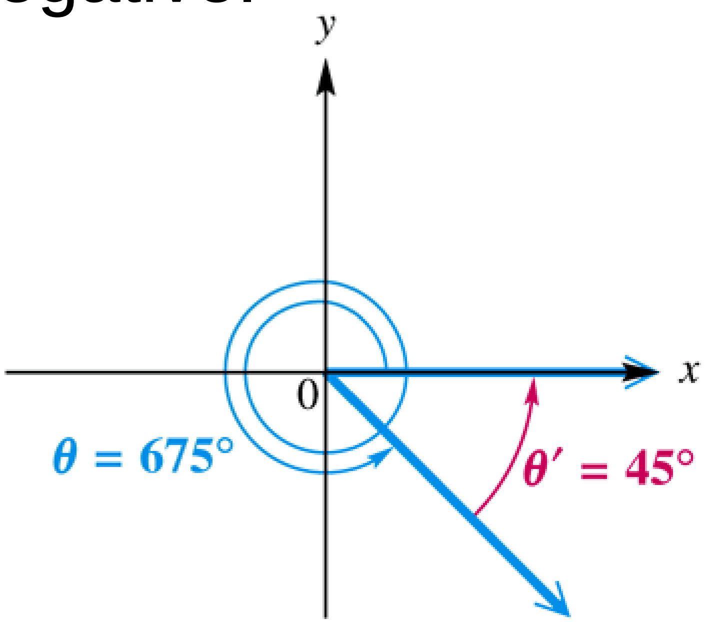
$$\cos(-240^\circ) = \cos 120^\circ$$
$$= -\cos 60^\circ$$
$$= -\frac{1}{2}$$

Example 3(b) FINDING TRIGONOMETRIC FUNCTION VALUES USING REFERENCE ANGLES

Find the exact value of tan 675°.

Subtract 360° to find a coterminal angle between 0° and 360°: 675° – 360° = 315°.

The reference angle is 360° – 315° = 45º. An angle of 315º is in quadrant IV, so the tangent will be negative.



$$\tan 675^\circ = \tan 315^\circ$$
$$= -\tan 45^\circ$$
$$= -1$$

Example 4 USING FUNCTION VALUES OF SPECIAL ANGLES

Evaluate $\cos 120^\circ + 2\sin^2 60^\circ - \tan^2 30^\circ$.

$$\cos 120^\circ = -\frac{1}{2} \qquad \sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{\sqrt{3}}{3}$$

$$\begin{aligned}\cos 120^\circ + 2\sin^2 60^\circ - \tan^2 30^\circ &= -\frac{1}{2} + 2\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{3}\right)^2 \\ &= -\frac{1}{2} + 2\left(\frac{3}{4}\right) - \frac{3}{9} \\ &= \frac{2}{3}\end{aligned}$$

Example 5(a) USING COTERMINAL ANGLES TO FIND FUNCTION VALUES

Evaluate cos 780° by first expressing the function in terms of an angle between 0° and 360°.

$$\cos 780^\circ = \cos(780^\circ - 2 \cdot 360^\circ)$$

$$= \cos 60^\circ$$

$$= \frac{1}{2}$$

Example 5(b) USING COTERMINAL ANGLES TO FIND FUNCTION VALUES

Evaluate cot(–405°) by first expressing the function in terms of an angle between 0° and 360°.

$\cot(-405°) = \cot(-405° + 2 \cdot 360°) = \cot 315°$.

The angle 315º is located in quadrant IV, and its reference angle is 45°.

$$\cot 315° = -\cot 45°$$

$$= -1$$

Example 6 FINDING ANGLE MEASURES GIVEN AN INTERVAL AND A FUNCTION VALUE

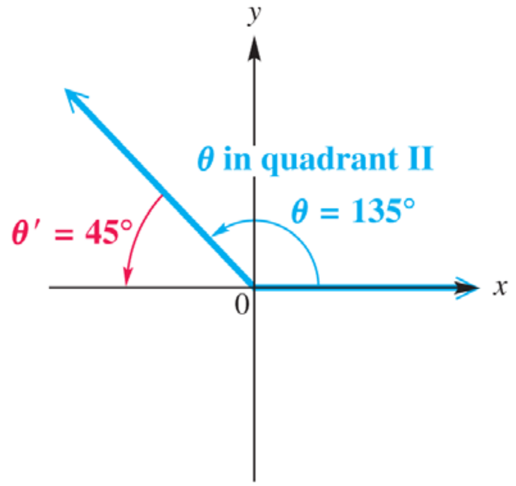
Find all values of θ, if θ is in the interval [0°, 360°) and

$$\cos\theta = -\frac{\sqrt{2}}{2}.$$

Since cos θ is negative, θ must lie in quadrant II or III.

The absolute value of cos θ is $\frac{\sqrt{2}}{2}$, so the reference angle is 45°.

The angle in quadrant II is 180° – 45° = 135°.



The angle in quadrant III is 180° + 45° = 225°.