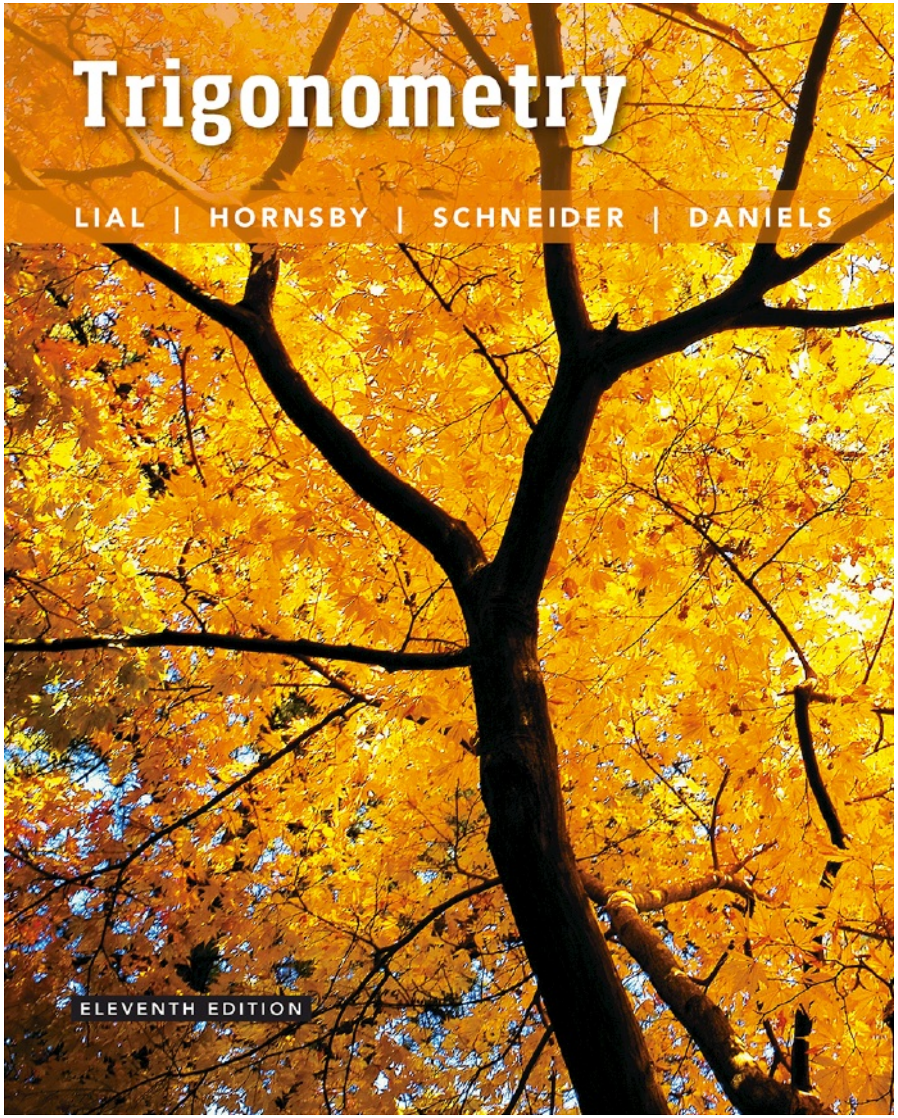
4

Graphs of the Circular Functions



4.1 Graphs of the Sine and Cosine Functions

Periodic Functions ▪ Graph of the Sine Function ▪ Graph of the Cosine Function ▪ Graphing Techniques, Amplitude, and Period ▪ Connecting Graphs with Equations ▪ A Trigonometric Model

Periodic Functions

Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight.

This periodic graph represents a normal heartbeat.

Periodic Function

A **periodic function** is a function f such that

$$f(x) = f(x + np),$$

for every real number x in the domain of f, every integer n, and some positive real number p. The least possible positive value of p is the **period** of the function.

Periodic Functions

The circumference of the unit circle is 2π, so the least possible value of *p* for which the sine and cosine functions repeat is 2π.

Therefore, the sine and cosine functions are periodic functions with period $\boldsymbol{2\pi}$ and the following statements are true for every integer *n*.

$$\sin x = \sin(x + n \bullet 2\pi) \quad \text{and} \quad \cos x = \cos(x + n \bullet 2\pi)$$

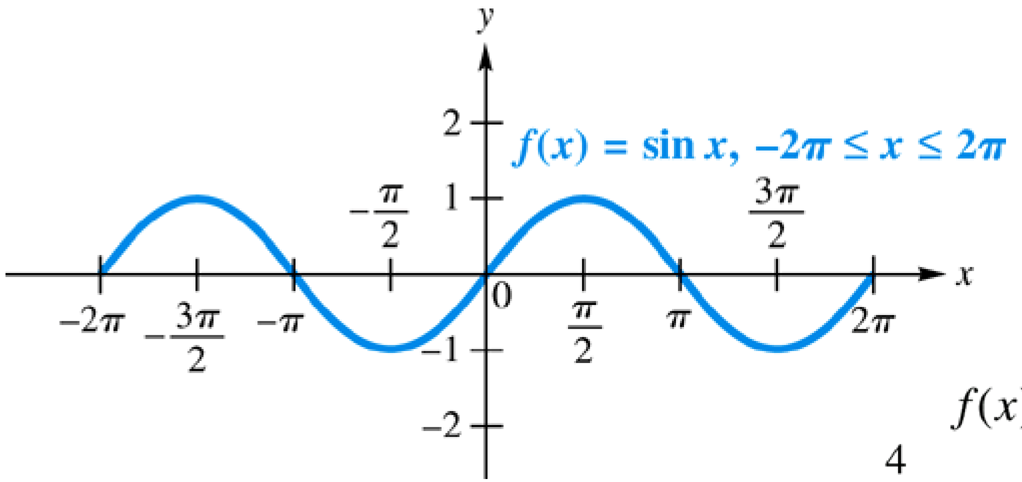
Values of the Sine and Cosine Functions

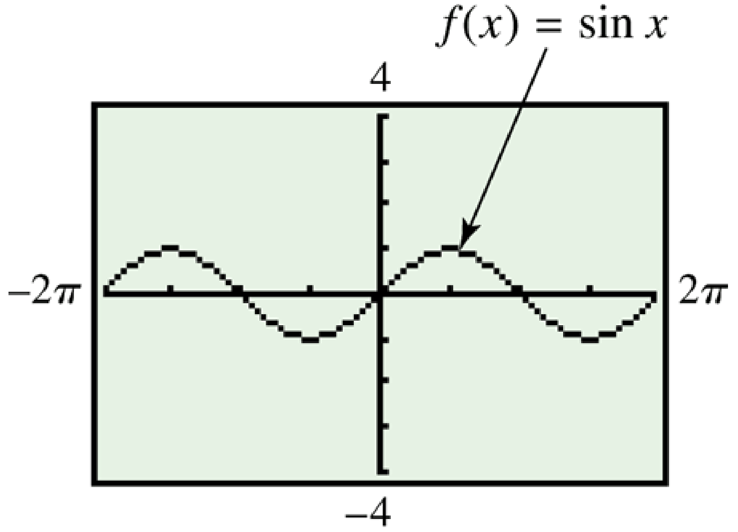
As s Increases from	$\sin s$	$\cos s$
0 to $\frac{\pi}{2}$	Increases from 0 to 1	Decreases from 1 to 0
$\frac{\pi}{2}$ to π	Decreases from 1 to 0	Decreases from 0 to -1
π to $\frac{3\pi}{2}$	Decreases from 0 to -1	Increases from -1 to 0
$\frac{3\pi}{2}$ to 2π	Increases from -1 to 0	Increases from 0 to 1

Sine Function $f(x) = \sin x$

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0





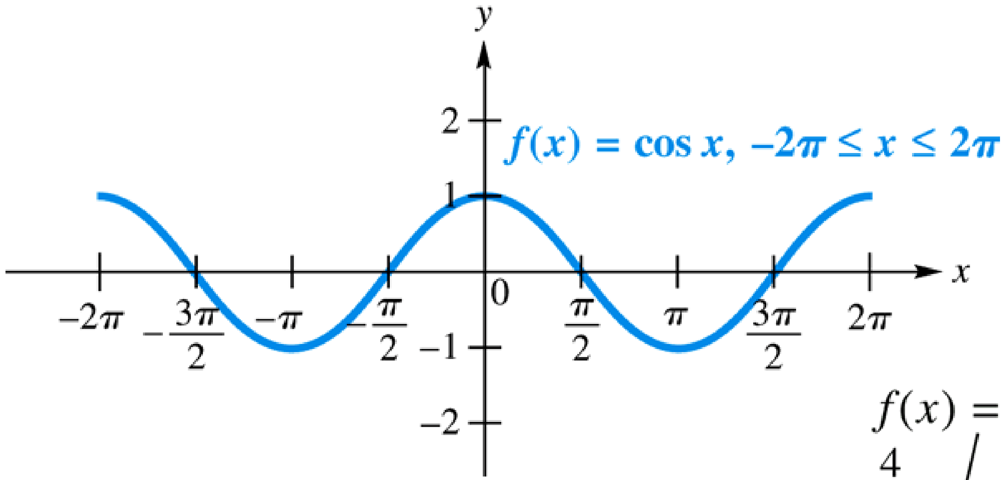
Sine Function $f(x) = \sin x$

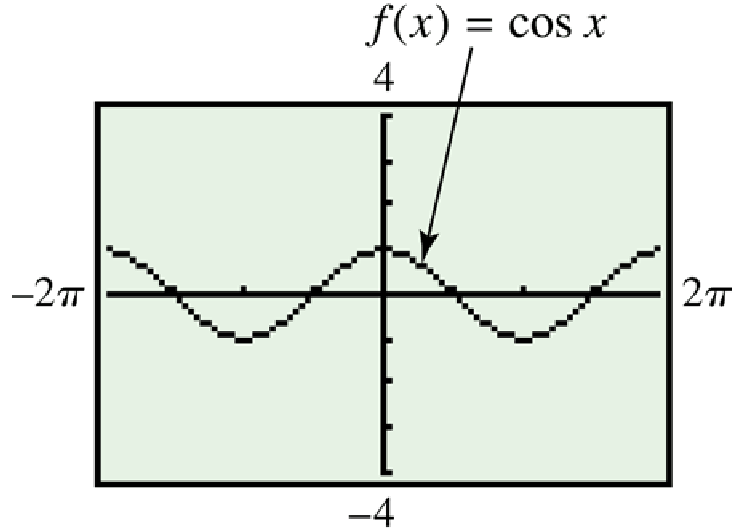
- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its x-intercepts are of the form $n\pi$, where n is an integer.
- Its period is 2π.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\sin(-x) = -\sin x$.

Cosine Function $f(x) = \cos x$

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

x	y
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1





Cosine Function $f(x) = \cos x$

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its x-intercepts are of the form $(2n+1)\frac{\pi}{2}$, where n is an integer.
- Its period is 2π.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $\cos(-x) = \cos x$.

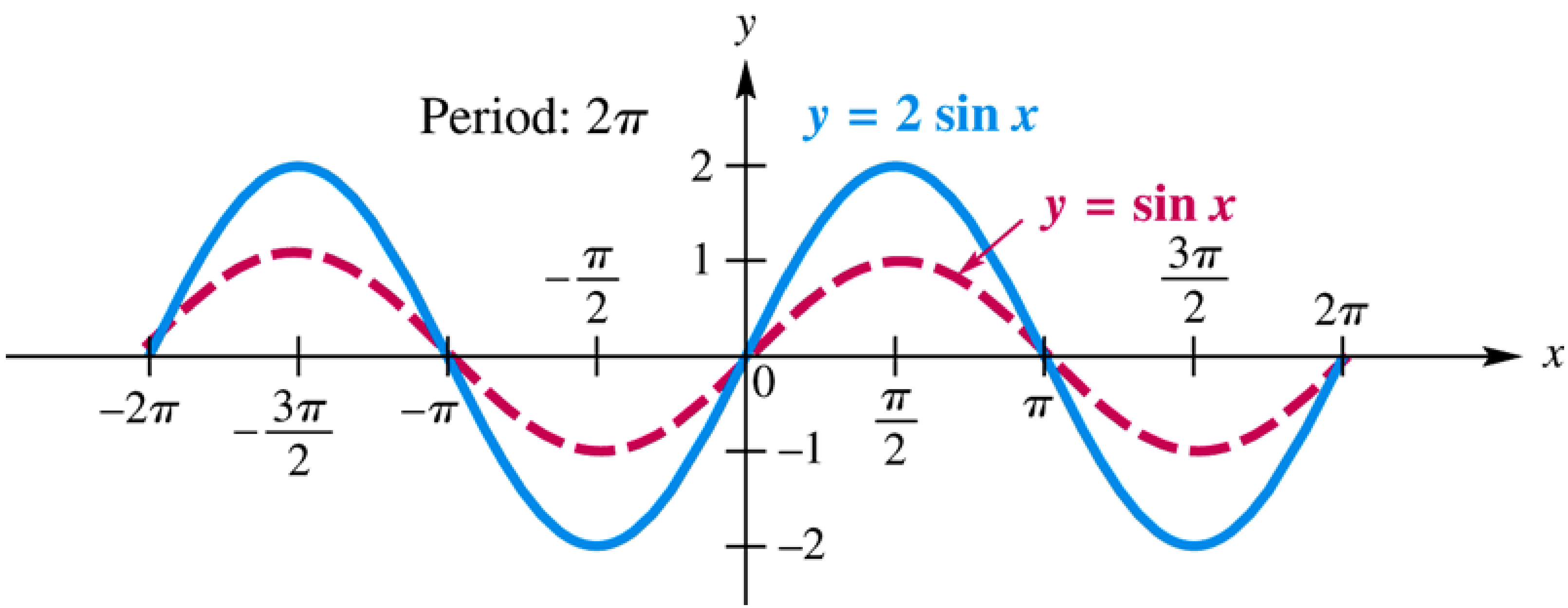
Example 1 GRAPHING $y = a \sin x$

Graph $y = 2\sin x$, and compare to the graph of $y = \sin x$.

For a given value of x, the value of y is twice what it would be for $y = \sin x$. The only change in the graph is the range, which becomes [–2, 2].

x	0	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	1	0	–1	0
$2 \sin x$	0	2	0	–2	0

Example 1 GRAPHING $y = a \sin x$ (continued)



Amplitude

The **amplitude** of a periodic function is half the difference between the maximum and minimum values. It describes the height of the graph both above and below a horizontal line passing through the "middle" of the graph.

Amplitude

The graph of ***y* = *a* sin *x*** or ***y* = *a* cos *x*,** with $a \neq 0$, will have the same shape as the graph of $y = \sin x$ or $y = \cos x$, respectively, except with range $[-|a|, |a|]$.

The amplitude is $|a|$.

Graphs of the Sine and Cosine Functions

No matter what the value of the amplitude, the periods of $y = a \sin x$ and $y = a \cos x$ are still 2π.

Now consider $y = \sin 2x$.

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\sin 2x$	0	1	0	–1	0	1	0	–1	0

One complete cycle occurs in π units.

Graphs of the Sine and Cosine Functions

Now consider $y = \sin 4x$.

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$	$7\pi/8$	π
$\sin 4x$	0	1	0	–1	0	1	0	–1	0

One complete cycle occurs in $\frac{\pi}{2}$ units.

In general, the graph of a function of the form $y = \sin bx$ or $y = \cos bx$, for $b > 0$, will have a period different from 2π when $b \neq 1$.

Graphs of the Sine and Cosine Functions

Divide the interval $\left[0, \frac{2\pi}{b}\right]$ into four equal parts to obtain the values for which sin *bx* or cos *bx* equal –1, 0, or 1.

These values give the minimum points, *x*-intercepts, and maximum points on the graph.

Find the midpoint of the interval by adding the *x*-values of the endpoints and dividing by 2. Then find the midpoints of these two intervals using the same procedure.

Example 2 GRAPHING $y = \sin bx$

Graph $y = \sin 2x$ and compare to the graph of $y = \sin x$.

The coefficient of x is 2, so $b = 2$, and the period is $\frac{2\pi}{2} = \pi$. The graph will complete one period over the interval $[0, \pi]$.

The endpoints are 0 and π, and the three points between the endpoints are

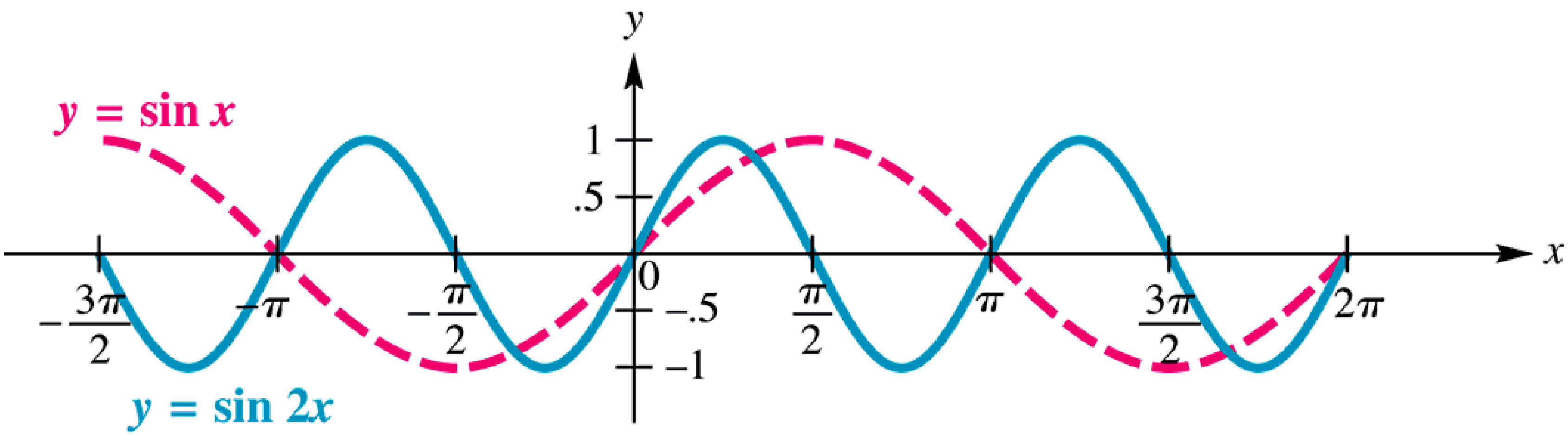
$$\frac{1}{4}(0+\pi) \qquad \frac{1}{2}(0+\pi), \qquad \text{and} \qquad \frac{3}{4}(0+\pi).$$

Example 2 GRAPHING $y = \sin bx$ (continued)

The x-values are

0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
↑	↑	↑	↑	↑
Left endpoint	First-quarter point	Midpoint	Third-quarter point	Right endpoint

▶ Example 2 GRAPHING *y* = sin *bx* (continued)



Period

For $b > 0$, the graph of **y = sin bx** will resemble that of y = sin x, but with period $\frac{2\pi}{b}$.

For $b > 0$, the graph of **y = cos bx** will resemble that of y = cos x, but with period $\frac{2\pi}{b}$.

Example 3 GRAPHING $y = \cos bx$

Graph $y = \cos \frac{2}{3} x$ over one period.

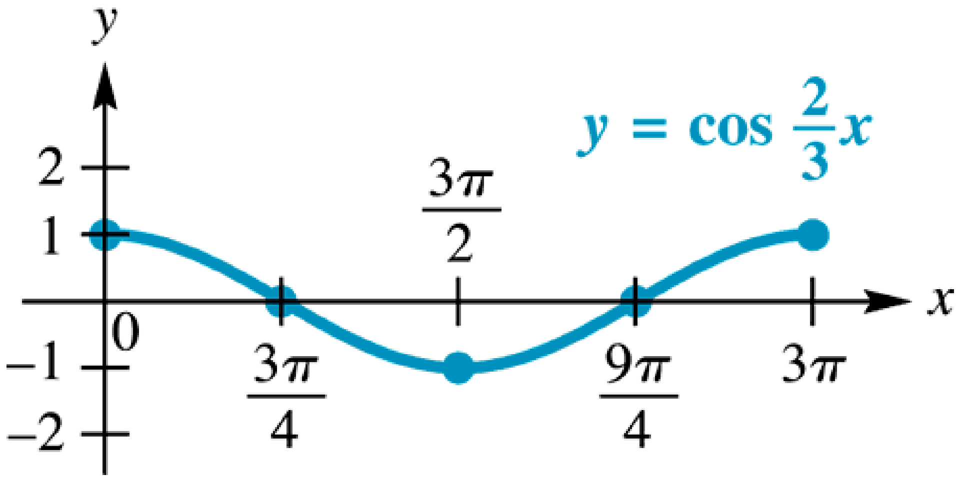
The period is $\frac{2\pi}{\frac{2}{3}} = 2\pi \cdot \frac{3}{2} = 3\pi.$

The endpoints are 0 and 3π, and the three points between the endpoints are $\frac{3\pi}{4}, \frac{3\pi}{2},$ and $\frac{9\pi}{4}.$

Example 3 GRAPHING $y = \cos bx$ (continued)

x	0	$3\pi/4$	$3\pi/2$	$9\pi/4$	3π
$2/3x$	0	$\pi/2$	π	$3\pi/2$	2π
$\cos 2/3x$	1	0	–1	0	1

The amplitude is 1.



Guidelines for Sketching Graphs of Sine and Cosine Functions

To graph $y = a \sin bx$ or $y = a \cos bx$, with $b > 0$, follow these steps.

Step 1 Find the period, $\frac{2\pi}{b}$. Start with 0 on the x-axis, and lay off a distance of $\frac{2\pi}{b}$.

Step 2 Divide the interval into four equal parts.

Step 3 Evaluate the function for each of the five x-values resulting from Step 2. The points will be maximum points, minimum points, and x-intercepts.

Guidelines for Sketching Graphs of Sine and Cosine Functions

Step 4 Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5 Draw the graph over additional periods as needed.

Example 4 GRAPHING $y = a \sin bx$

Graph $y = -2 \sin 3x$ over one period.

Step 1

For this function, $b = 3$, so the period is $\frac{2\pi}{3}$.

The function will be graphed over the interval $\left[0, \frac{2\pi}{3}\right]$.

Step 2

Divide the interval $\left[0, \frac{2\pi}{3}\right]$ into four equal parts to get the x-values $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2},$ and $\frac{2\pi}{3}$.

Example 4 GRAPHING $y = a \sin bx$ (continued)

Step 3

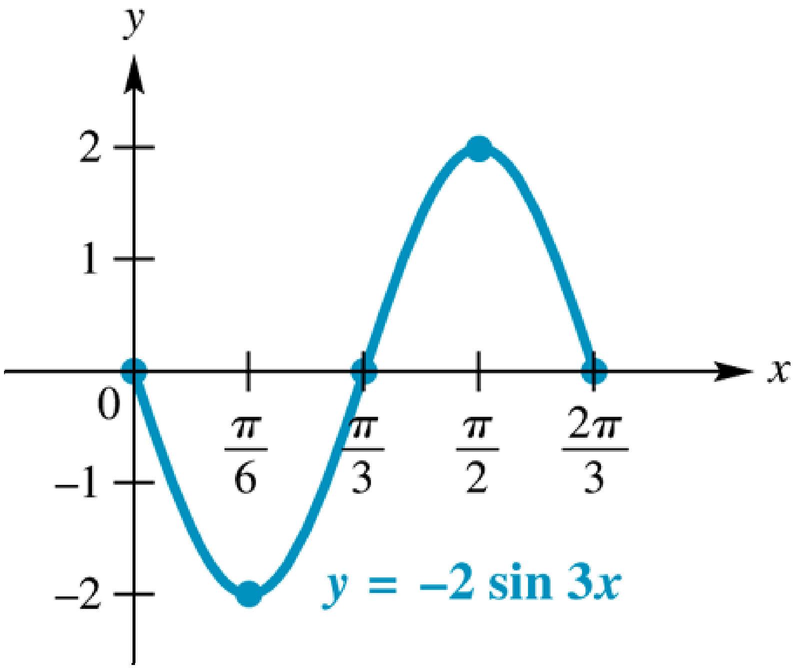
Make a table of values determined by the *x*-values from Step 2.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$
$3x$	0	$\pi/2$	π	$3\pi/2$	2π
$\sin 3x$	0	1	0	–1	0
$-2 \sin 3x$	0	–2	0	0	0

Steps 4, 5

Plot the points $(0,0), \left(\frac{\pi}{6},-2\right), \left(\frac{\pi}{3},0\right), \left(\frac{\pi}{2},2\right)$, and $\left(\frac{2\pi}{3},0\right)$.

Join the points with a sinusoidal curve with amplitude 2. The graph can be extended by repeating the cycle.



Note

When a is negative, the graph of $y = a \sin bx$ is the reflection across the x-axis of the graph of $y = |a| \sin bx$.

Example 5 GRAPHING $y = a \cos bx$ FOR b THAT IS A MULTIPLE OF π

Graph $y = -3 \cos \pi x$ over one period.

Step 1

Since $b = \pi$, the period is $\frac{2\pi}{\pi} = 2$.

The function will be graphed over the interval [0, 2].

Step 2

Divide the interval [0, 2] into four equal parts to get the *x*-values $0, \frac{1}{2}, 1, \frac{3}{2}$, and 2.

Example 5 GRAPHING $y = a \cos bx$ FOR b THAT IS A MULTIPLE OF π (continued)

Step 3

Make a table of values determined by the *x*-values from Step 2.

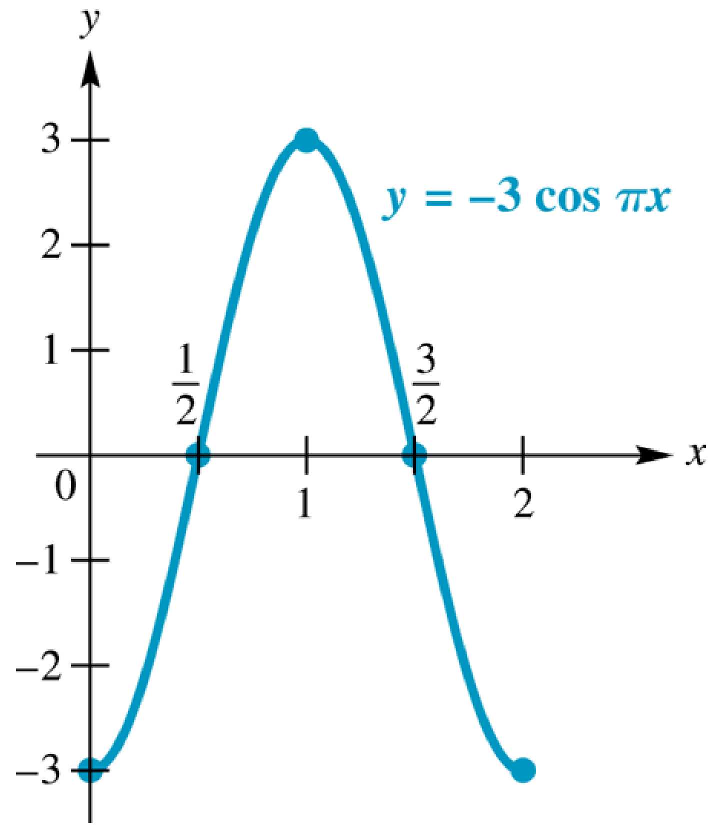
x	0	1/2	1	3/2	2
πx	0	$\pi/2$	π	$3\pi/2$	2π
$\cos \pi x$	1	0	–1	0	1
$-3 \cos \pi x$	–3	0	3	0	–3

Example 5 GRAPHING $y = a \cos bx$ FOR b THAT IS A MULTIPLE OF π (continued)

Steps 4, 5

Plot the points $(0,-3)$, $\left(\frac{1}{2},0\right)$, $(1,3)$, $\left(\frac{3}{2},0\right)$, and $(2,-3)$.

Join the points with a sinusoidal curve with amplitude $|-3| = 3$. The graph can be extended by repeating the cycle.

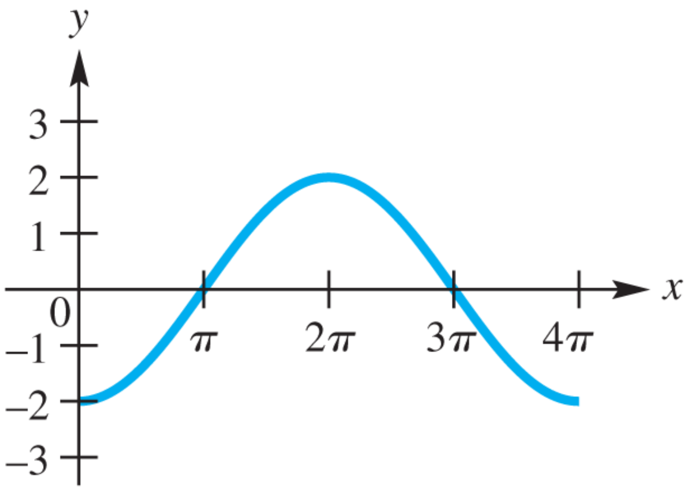


Note

When b is an integer multiple of π, the x-intercepts of the graph are rational numbers.

Example 6 DETERMINING AN EQUATION FOR A GRAPH

Determine an equation of the form $y = a \cos bx$ or $y = a \sin bx$, where $b > 0$, for the given graph.

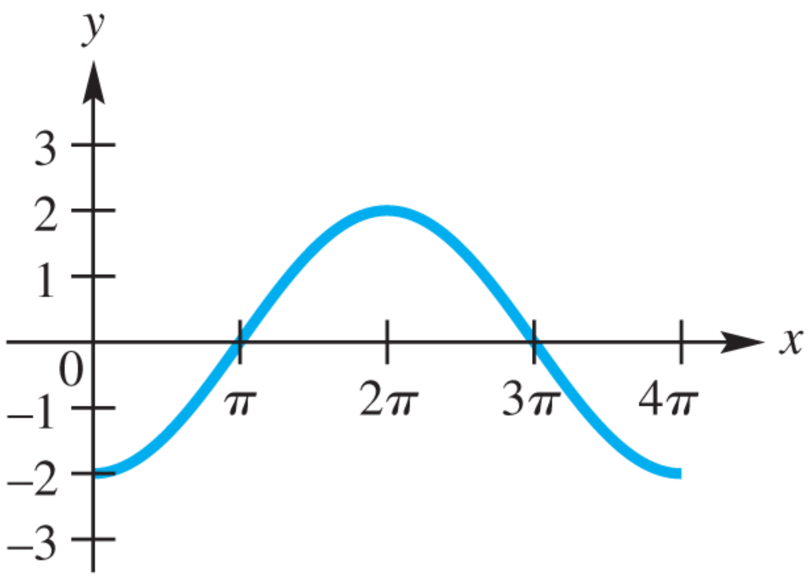


This graph is that of a cosine function that is reflected across its horizontal axis, the *x*-axis. The amplitude is half the distance between the maximum and minimum values.

$$\frac{1}{2}\left[2-(-2)\right]=\frac{1}{2}(4)=2$$

Example 6 DETERMINING AN EQUATION FOR A GRAPH (continued)

Because the graph complete a cycle on the interval $[0, 4\pi]$, the period is 4π. We use this fact to solve for *b*.



$$4\pi = \frac{2\pi}{b}$$

$$4\pi b = 2\pi$$

$$b = \frac{1}{2}$$

An equation for the graph is

$$y = -2\cos\frac{1}{2}x.$$

Example 7 INTERPRETING A SINE FUNCTION MODEL

The average temperature (in °F) at Mould Bay, Canada, can be approximated by the function

$$f(x) = 34\sin\left[\frac{\pi}{6}(x - 4.3)\right]$$

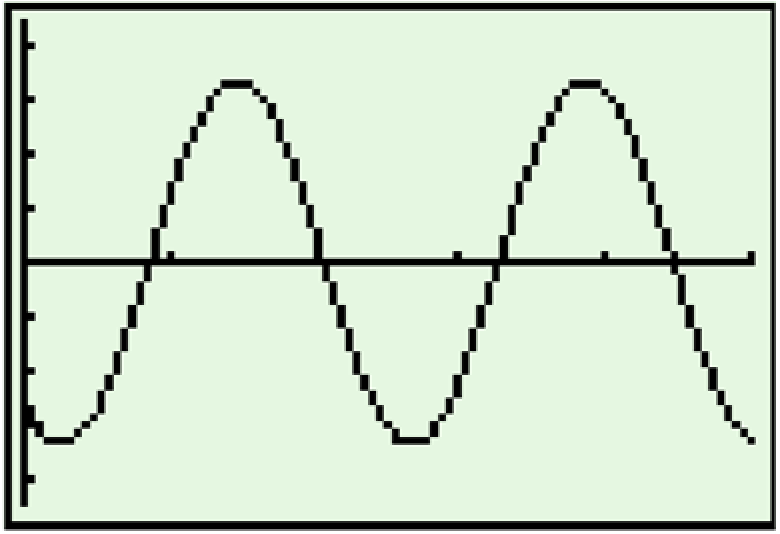
where x is the month and $x = 1$ corresponds to January, $x = 2$ corresponds to February, and so on.

(a) To observe the graph over a two-year interval and to see the maximum and minimum points, graph f in the window [0, 25] by [–45, 45].

Example 7 INTERPRETING A SINE FUNCTION MODEL (continued)

The amplitude of the graph is 34 and the period is

$$\frac{2\pi}{\frac{\pi}{6}} = 12.$$



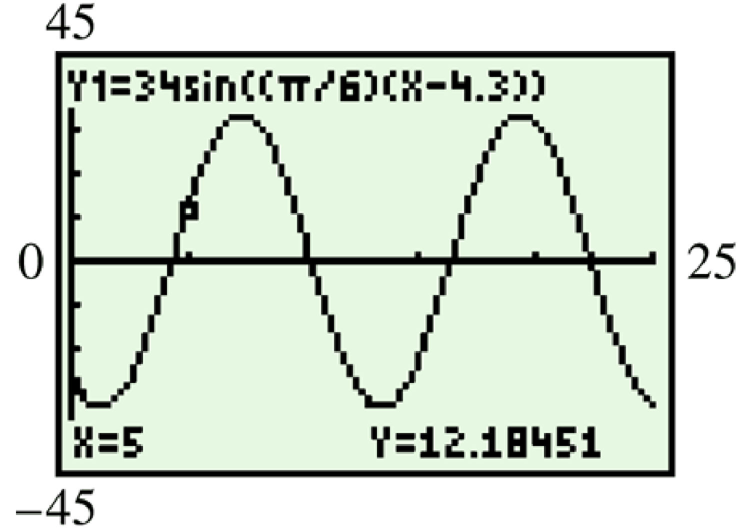
The function has a period of 12 months, or 1 year, which agrees with the changing of the seasons.

Example 7 INTERPRETING A SINE FUNCTION MODEL (continued)

(b) According to this model, what is the average temperature during the month of May?

May is the fifth month, so the average temperature during May is

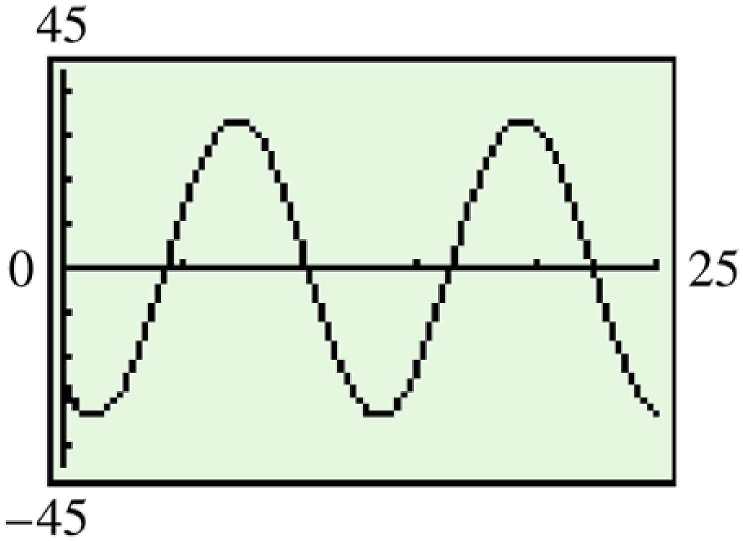
$$f(5) = 34\sin\left[\frac{\pi}{6}(5-4.3)\right] \approx 12.18$$



See the display at the bottom of the screen.

Example 7 INTERPRETING A SINE FUNCTION MODEL (continued)

(c) What would be an approximation for the average *yearly* temperature at Mould Bay?



From the graph, it appears that the average yearly temperature is about 0°F since the graph is centered vertically about the line $y = 0$.