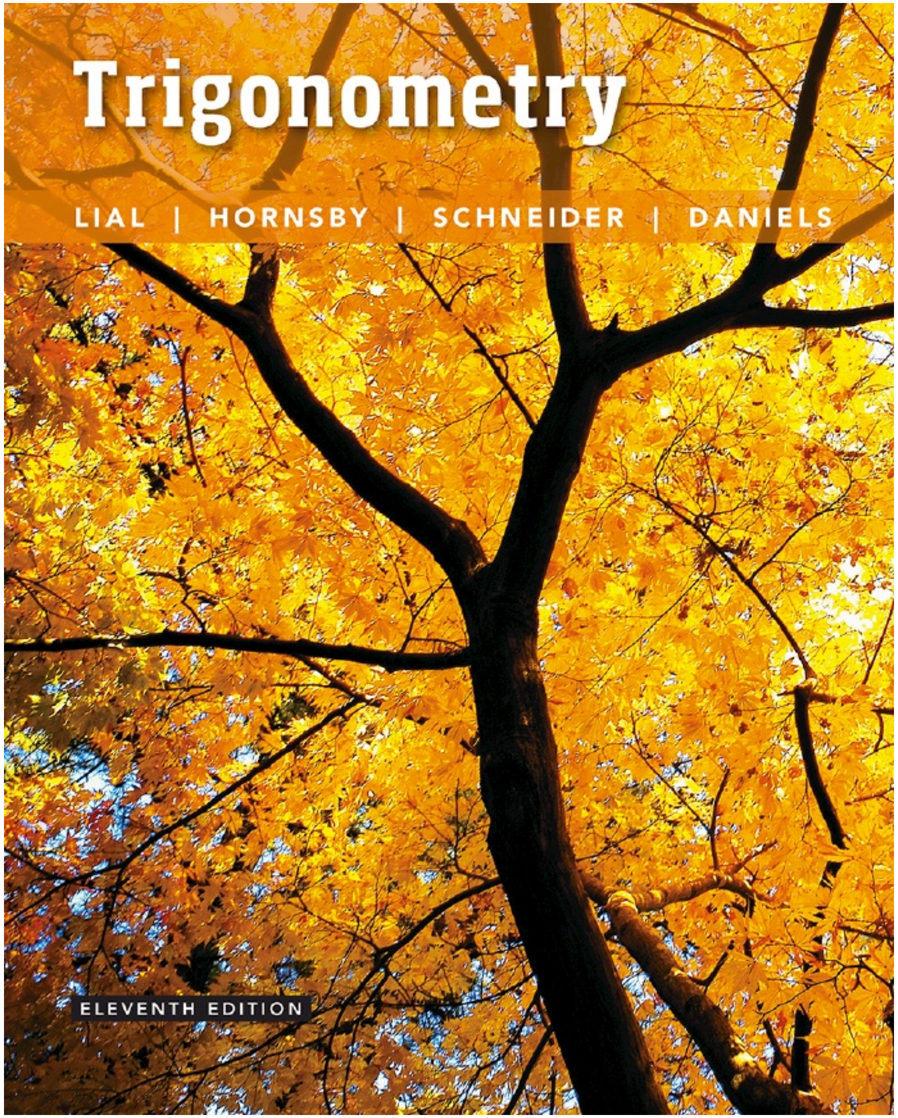
4

Graphs of the Circular Functions

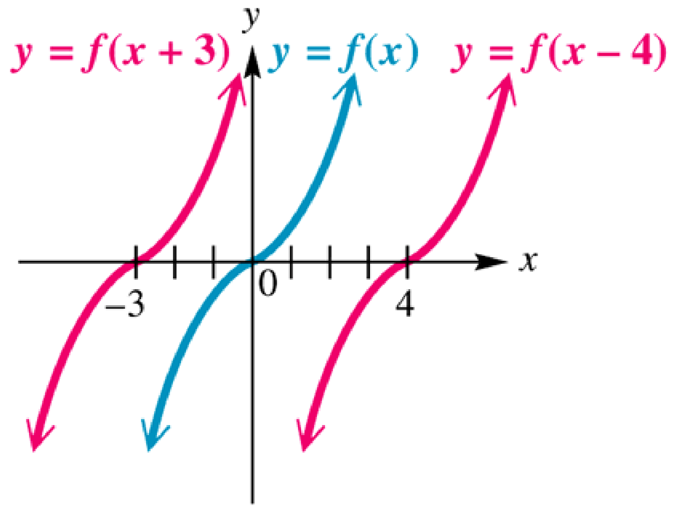


4.2 Translations of the Graphs of the Sine and Cosine Functions

Horizontal Translations ▪ Vertical Translations ▪ Combinations of Translations ▪ A Trigonometric Model

Horizontal Translations

The graph of the function $y = f(x - d)$ is translated *horizontally* compared to the graph of $y = f(x)$.



Horizontal translations of $y = f(x)$

The translation is d units to the right if $d > 0$ and is $|d|$ units to the left if $d < 0$.

Horizontal Translations

With circular functions, a horizontal translation is called a ***phase shift.***

In the function $y = f(x - d)$, the expression $x - d$ is the ***argument***.

Example 1 GRAPHING $y = \sin(x - d)$

Graph $y = \sin\left(x - \frac{\pi}{3}\right)$ over one period.

Method 1

To find an interval of one period, solve the three-part inequality $0 \le x - \frac{\pi}{3} \le 2\pi \Rightarrow \frac{\pi}{3} \le x \le \frac{7\pi}{3}.$

Divide this interval into four equal parts:

$$\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}$$

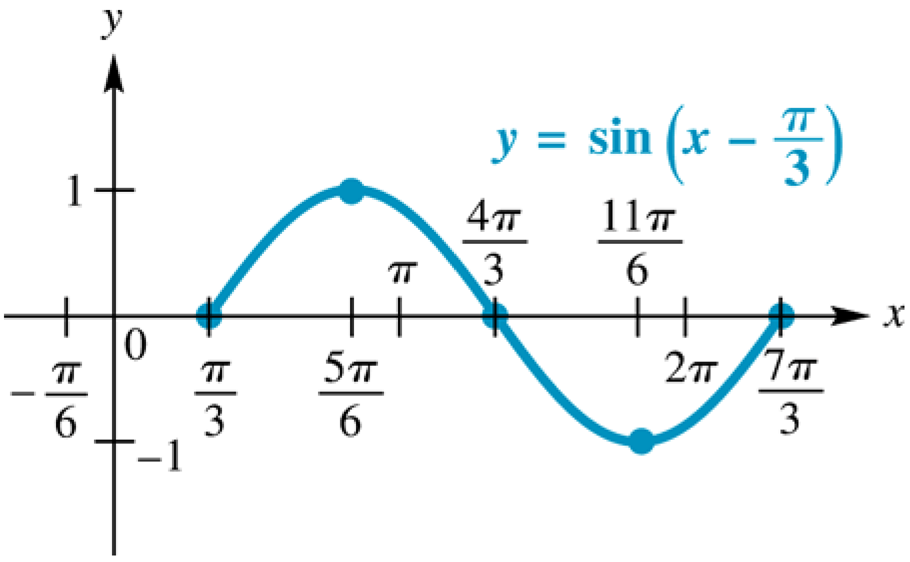
Example 1 GRAPHING $y = \sin(x - d)$ (continued)

Make a table of values determined by the *x*-values.

x	$\pi/3$	$5\pi/6$	$4\pi/3$	$11\pi/6$	$7\pi/3$
$x - \pi/3$	0	$\pi/2$	π	$3\pi/2$	2π
$\sin(x - \pi/3)$	0	1	0	–1	0

Example 1 GRAPHING $y = \sin(x - d)$ (continued)

Join the corresponding points with a smooth curve.

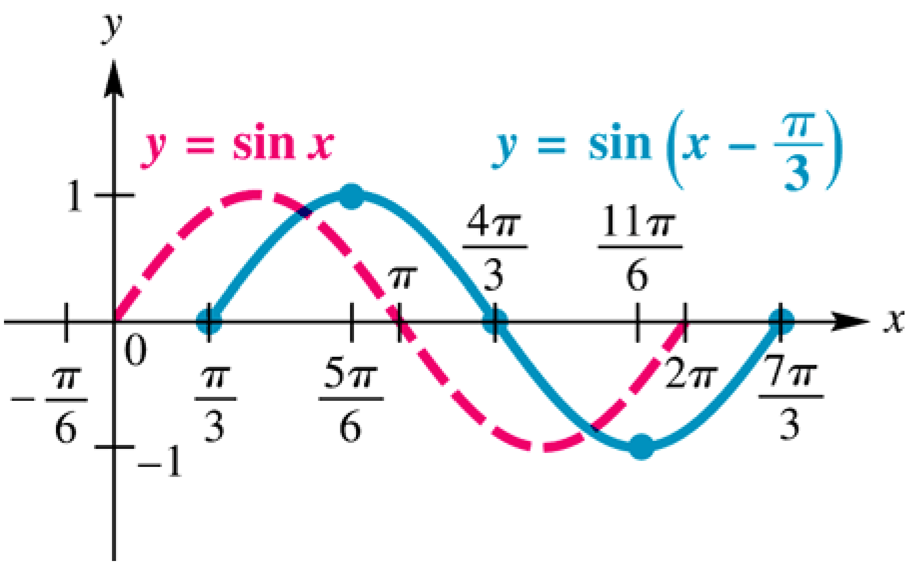


The period is 2π, and the amplitude is 1.

Example 1 GRAPHING $y = \sin(x - d)$ (continued)

Method 2

The argument $x - \frac{\pi}{3}$ indicates that the graph of $y = \sin x$ will be translated $\frac{\pi}{3}$ units to the right.



Example 2 GRAPHING $y = a \cos(x - d)$

Graph $y = 3\cos\left(x + \frac{\pi}{4}\right)$ over one period.

Method 1

To find an interval of one period, solve the three-part inequality $0 \le x + \frac{\pi}{4} \le 2\pi \Rightarrow -\frac{\pi}{4} \le x \le \frac{7\pi}{4}.$

Divide this interval into four equal parts:

$$-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

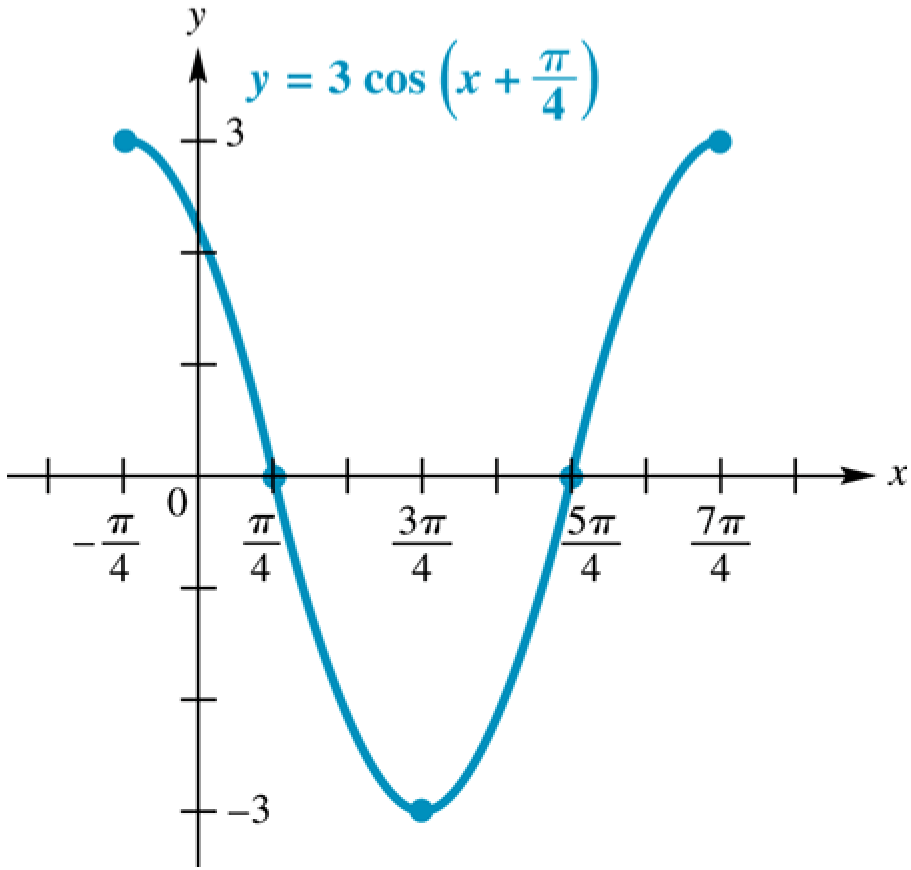
Example 2 GRAPHING $y = a \cos(x - d)$ (continued)

Make a table of values determined by the *x*-values.

x	$-\pi/4$	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$
$x + \pi/4$	0	$\pi/2$	π	$3\pi/4$	2π
$\cos(x + \pi/4)$	1	0	–1	0	1
$3\cos(x + \pi/4)$	3	0	–3	0	3

Example 2 GRAPHING $y = a \cos(x - d)$ (continued)

Join the corresponding points with a smooth curve.



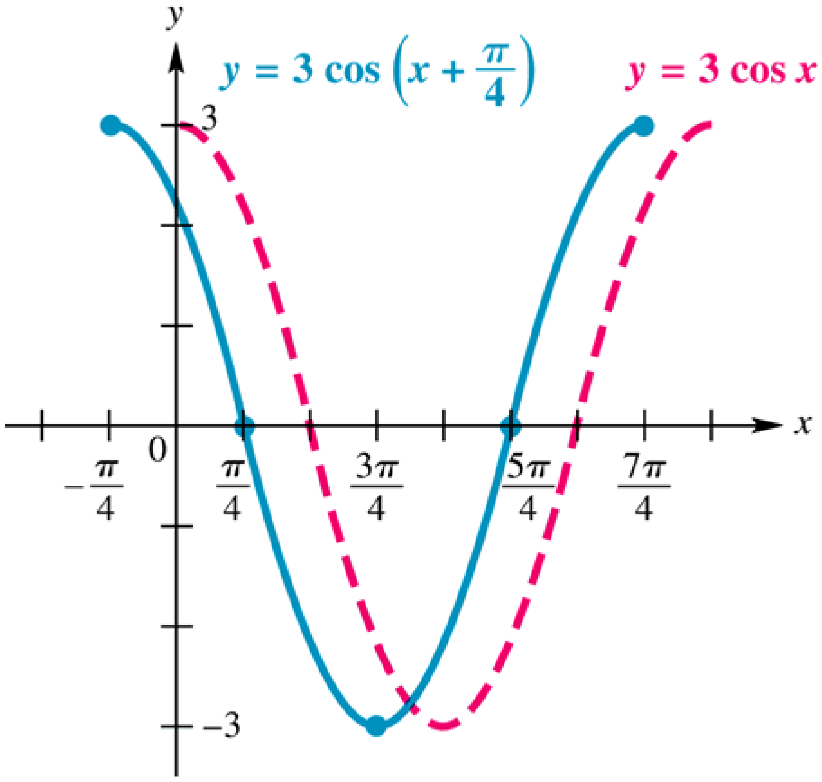
The period is 2π, and the amplitude is 3.

Example 2 GRAPHING $y = a\cos(x - d)$ (continued)

Method 2

$$3\cos\left(x + \frac{\pi}{4}\right) = 3\cos\left[x - \left(-\frac{\pi}{4}\right)\right]$$

$d = -\frac{\pi}{4}$, so the phase shift is $\frac{\pi}{4}$ unit to the left.



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Graph $y = -2\cos(3x + \pi)$ over two periods.

Method 1

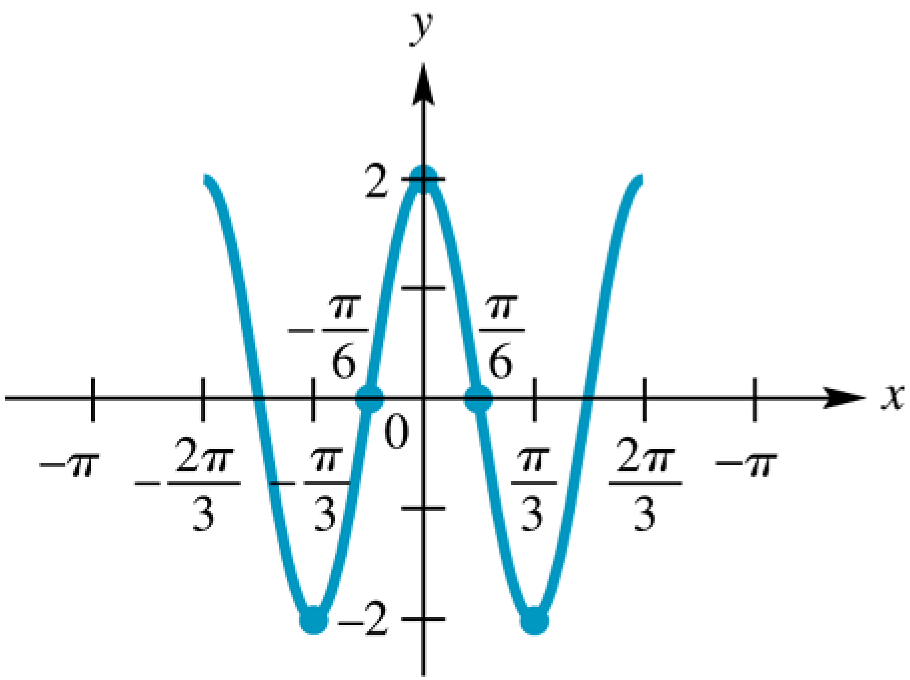
To find an interval of one period, solve the three-part inequality $0 \le 3x + \pi \le 2\pi \Rightarrow -\frac{\pi}{3} \le x \le \frac{\pi}{3}$.

Divide this interval into four equal parts to find the points

$$\left(-\frac{\pi}{3}, -2\right), \left(-\frac{\pi}{6}, 0\right), (0, 2), \left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{3}, -2\right).$$

▶ Example 3 GRAPHING $y = a \cos [b(x - d)]$ (continued)

Plot these points and join them with a smooth curve. Then graph an additional half-period to the left and to the right.



$y = -2 \cos (3x + \pi)$

Example 3 GRAPHING $y = a\cos[b(x - d)]$ (continued)

Method 2

Write the expression in the form $\boldsymbol{a\cos[b(x - d)]}$.

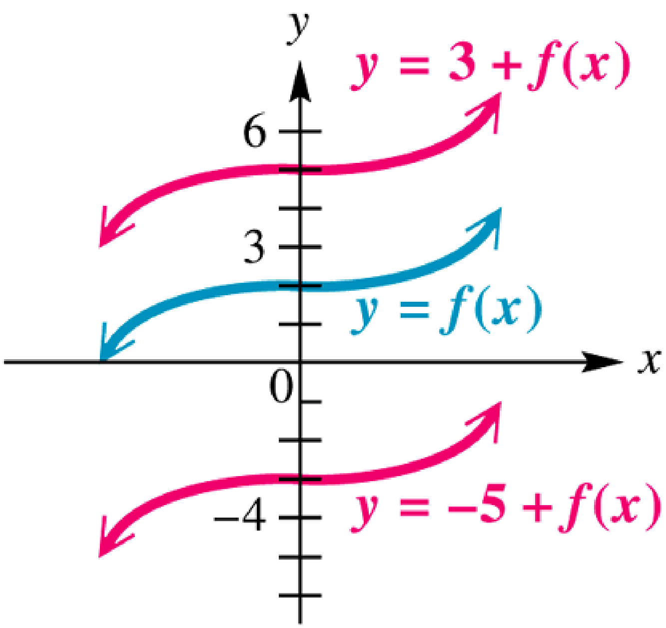
$$y = -2\cos(3x + \pi) = -2\cos\left[3\left(x + \frac{\pi}{3}\right)\right]$$

Then $a = -2$, $b = 3$, and $d = -\frac{\pi}{3}$.

The amplitude is $|-2| = 2$, the period is $\frac{2\pi}{3}$, and the phase shift is $\frac{\pi}{3}$ units to the left as compared to the graph of $y = -2\cos 3x$.

Vertical Translations

The graph of a function of the form ***y* = *c* + *f*(*x*)** is translated *vertically* compared to the graph of *y* = *f*(*x*).



Vertical translations of $y = f(x)$

The translation is *c* units up if *c* > 0 and |*c*| units down if *c* < 0.

Example 4 **GRAPHING $y = c + a \cos bx$**

Graph $y = 3 - 2 \cos 3x$ over two periods.

The graph of $y = 3 - 2 \cos 3x$ is the same as the graph of $y = -2 \cos 3x$, vertically translated 3 units up.

The period of $-2 \cos 3x$ is $\frac{2\pi}{3}$, so the key points have x-values

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}.$$

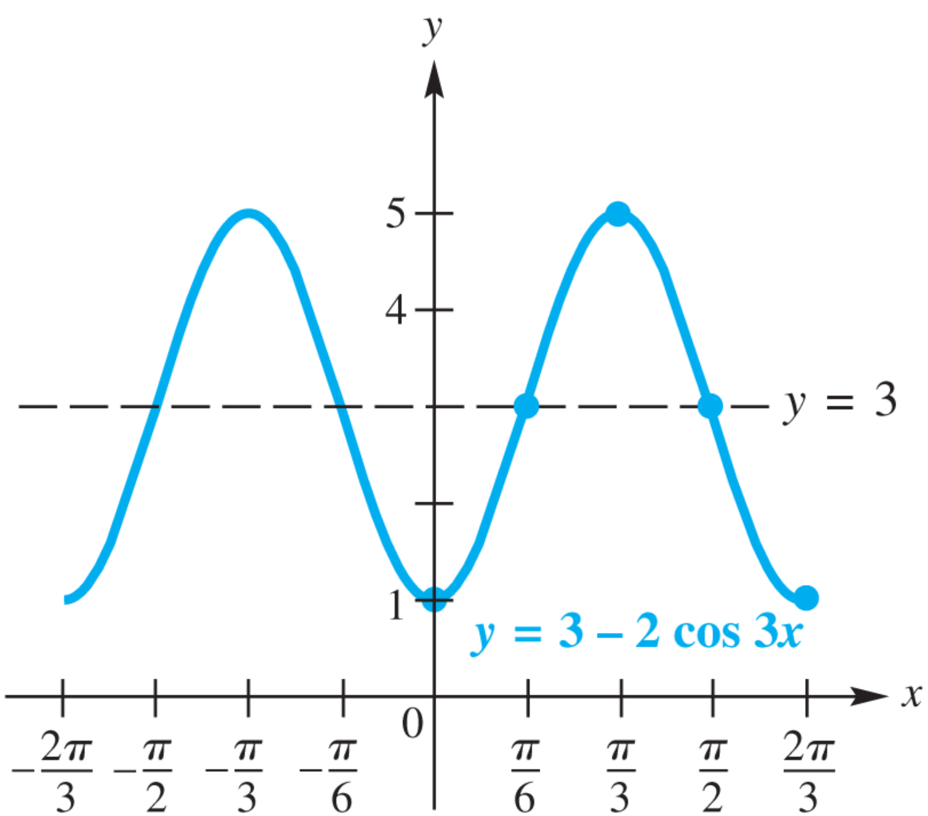
Example 4 GRAPHING $y = c + a \cos bx$ (continued)

Make a table of points.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$
$\cos 3x$	1	0	–1	0	1
$2 \cos 3x$	2	0	–2	0	2
$3 - 2 \cos 3x$	1	3	5	3	1

Example 4 GRAPHING $y = c + a \cos bx$ (continued)

The key points are shown on the graph , along with more of the graph, which is sketched using the fact that the function is periodic.



Further Guidelines for Sketching Graphs of Sine and Cosine Functions

Method 1

Step 1 Find an interval whose length is one period $\frac{2\pi}{b}$ by solving the three-part inequality $0 \leq b(x - d) \leq 2\pi$.

Step 2 Divide the interval into four equal parts.

Step 3 Evaluate the function for each of the five *x*-values resulting from Step 2. The points will be maximum points, minimum points, and points that intersect the line $y = c$.

Further Guidelines for Sketching Graphs of Sine and Cosine Functions

Step 4 Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5 Draw the graph over additional periods, as needed.

Further Guidelines for Sketching Graphs of Sine and Cosine Functions

Method 2

Step 1 Graph $y = a \sin bx$ or $y = a \cos bx$. The amplitude of the function is $|a|$, and the period is $\frac{2\pi}{b}$.

Step 2 Use translations to graph the desired function. The vertical translation is c units up if $c > 0$ and $|c|$ units down if $c < 0$. The horizontal translation (phase shift) is d units to the right if $d > 0$ and $|d|$ units to the left if $d < 0$.

Example 5 GRAPHING $y = c + a \sin[b(x - d)]$

Graph $y = -1 + 2\sin(4x + \pi)$ over two periods.

Use Method 1:

$$y = -1 + 2\sin(4x + \pi) \Rightarrow y = -1 + 2\sin\left[4\left(x + \frac{\pi}{4}\right)\right]$$

Step 1: Find an interval whose length is one period.

$$0 \le 4\left(x + \frac{\pi}{4}\right) \le 2\pi$$

$$0 \le x + \frac{\pi}{4} \le \frac{\pi}{2}$$ Divide each part by 4.

$$-\frac{\pi}{4} \le x \le \frac{\pi}{4}$$ Subtract $\frac{\pi}{4}$ from each part.

Step 2: Divide the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ into four equal parts to get these *x*-values.

$$-\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \frac{\pi}{4}$$

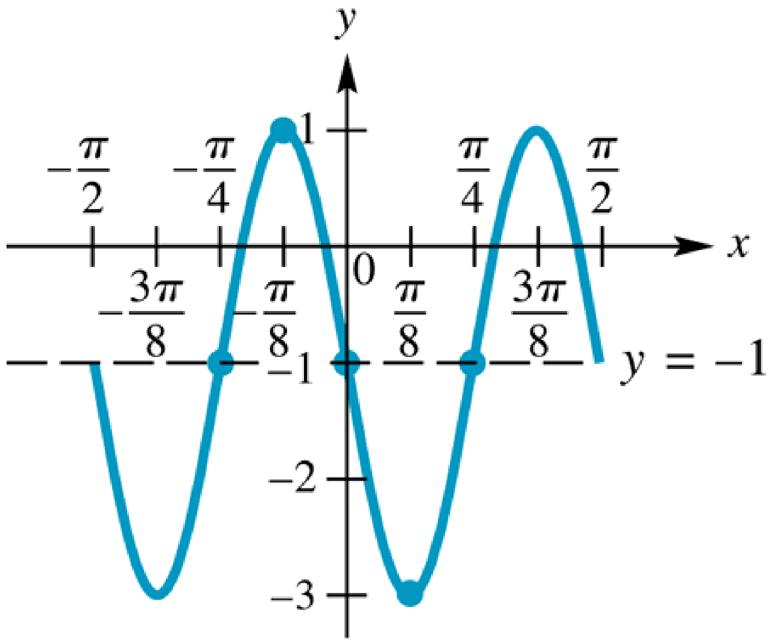
Step 3: Make a table of values.

Example 5 GRAPHING $y = c + a \sin [b(x - d)]$ (cont.)

x	$-\pi/4$	$-\pi/8$	0	$\pi/8$	$\pi/4$
$x + \pi/4$	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$4(x + \pi/4)$	2	$\pi/2$	π	$3\pi/2$	2π
$\sin [4(x + \pi/4)]$	0	1	0	-1	0
$2 \sin [4(x + \pi/4)]$	0	2	0	-2	0
$-1 + 2 \sin(4x + \pi)$	-1	1	-1	-3	-1

Example 5 GRAPHING $y = c + a \sin[b(x - d)]$ (cont.)

Steps 4 and 5: Plot the points found in the table and join them with a sinusoidal curve. Extend the graph to the right and to the left to include two full periods.



$y = -1 + 2\sin(4x + \pi)$

Example 6 MODELING TEMPERATURE WITH A SINE FUNCTION

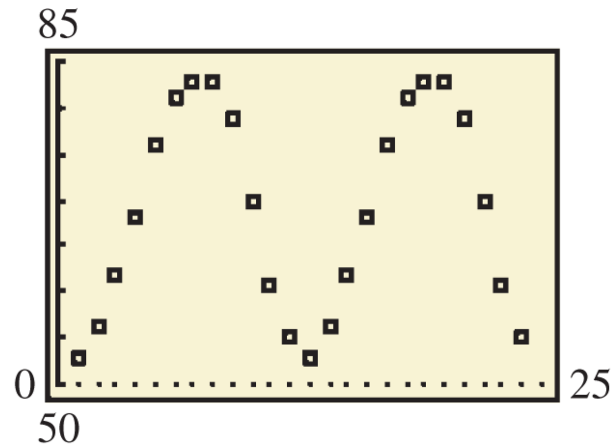
The maximum average monthly temperature in New Orleans is 83°F and the minimum is 53°F. The table shows the average monthly temperatures.

Month	°F	Month	°F
Jan	53	July	83
Feb	56	Aug	83
Mar	62	Sept	79
Apr	68	Oct	70
May	76	Nov	61
June	81	Dec	55

Source: World Almanac and Book of Facts.

Example 6 MODELING TEMPERATURE WITH A SINE FUNCTION (continued)

The scatter diagram for a two-year interval strongly suggests that the temperatures can be modeled with a sine curve.



(a) Using only the maximum and minimum temperatures, determine a function of the form

$$f(x) = a \sin\left[b(x - d)\right] + c,$$

where *a*, *b*, *c*, and *d* are constants, that models the average monthly temperature in New Orleans. Let *x* represent the month, with January corresponding to $x = 1$.

Example 6 MODELING TEMPERATURE WITH A SINE FUNCTION (continued)

Use the maximum and minimum average monthly temperatures to find the amplitude *a*.

$$a = \frac{83 - 53}{2} = 15$$

The average of the maximum and minimum temperatures is a good choice for *c*. The average is

$$\frac{83 + 53}{2} = 68.$$

Since temperatures repeat every 12 months, $b = \frac{2\pi}{12} = \frac{\pi}{6}$.

Example 6 MODELING TEMPERATURE WITH A SINE FUNCTION (continued)

To determine the phase shift, observe that the coldest month is January, when $x = 1$, and the hottest month is July, when $x = 7$. A good choice for d is 4, because April, when $x = 4$, is located at the midpoint between January and July. Also, notice that the average monthly temperature in April is 68ºF, which is the value of the vertical translation, c. The average monthly temperature in New Orleans is modeled closely by the following equation.

$$f(x) = a\sin[b(x-d)] + c = 15\sin\left[\frac{\pi}{6}(x-4)\right] + 68$$

▶ Example 6 MODELING TEMPERATURE WITH A SINE FUNCTION (continued)

(b) On the same coordinate axes, graph *f* for a two-year period together with the actual data values found in the table.

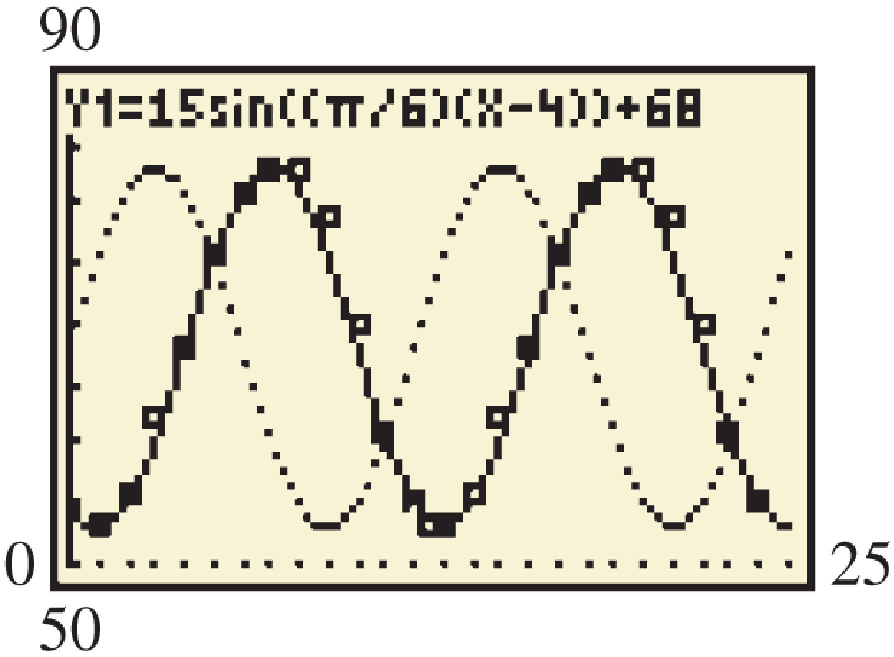
The figure shows the data points from the table, along with the graph of

$$y = 15\sin\left[\frac{\pi}{6}(x-4)\right] + 68$$

and the graph of

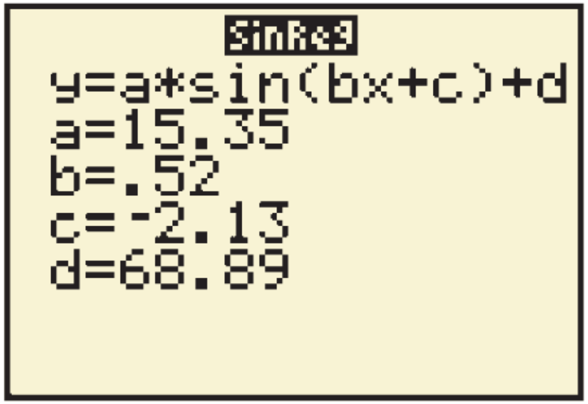
$$y = 15\sin\frac{\pi}{6}x + 68$$

for comparison.



(c) Use the **sine regression** feature of a graphing calculator to determine a second model for these data.

We used the given data for a two-year period and the sine regression capability of a graphing calculator to produce the model $f(x) = 15.35 \sin(0.52x - 2.13) + 68.89$.



Values are rounded to the nearest hundredth.