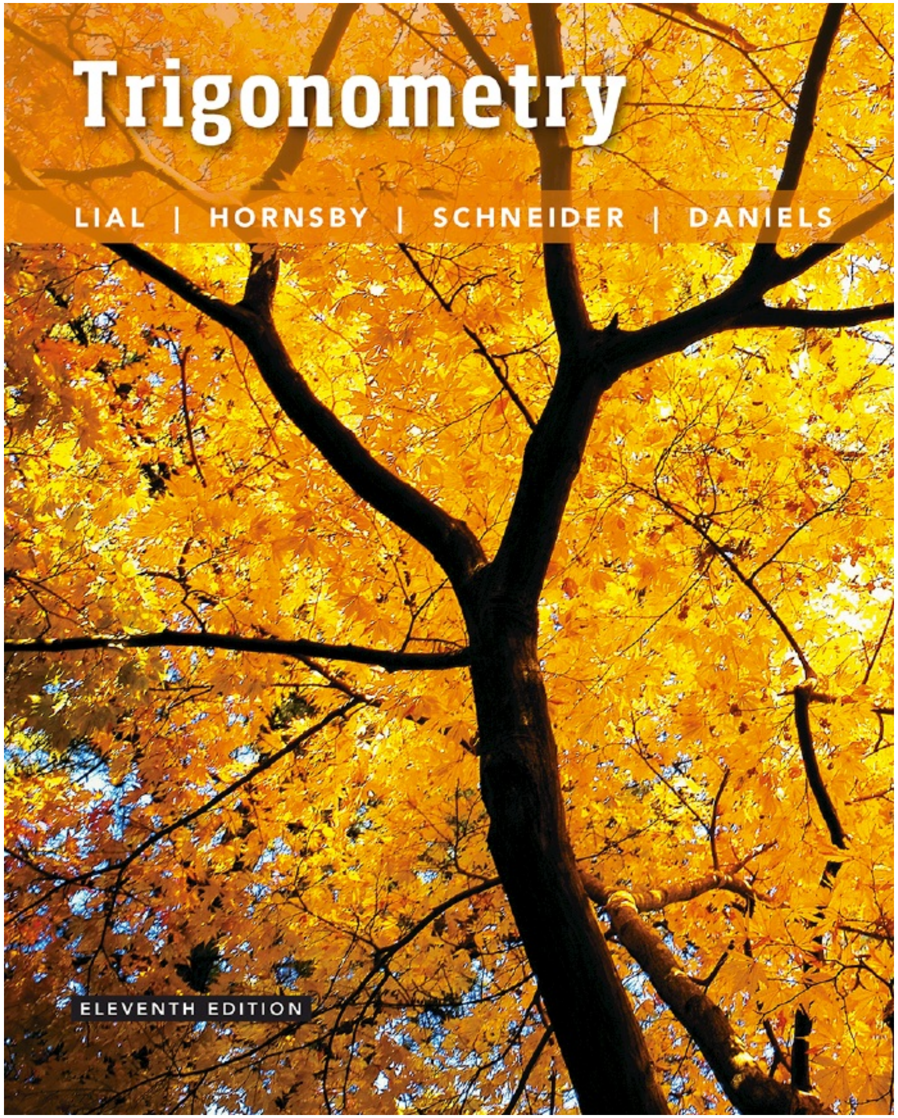
5

Trigonometric Identities



5.1 Fundamental Identities

Fundamental Identities ▪ Uses of the Fundamental Identities

Fundamental Identities

Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Fundamental Identities

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

▶ Note

In trigonometric identities, θ can be an angle in degrees, a real number, or a variable.

Example 1 FINDING TRIGONOMETRIC FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT

If $\tan\theta = -\frac{5}{3}$ and θ is in quadrant II, find each function value.

(a) $\sec\theta$

$$\tan^2\theta + 1 = \sec^2\theta$$

Pythagorean identity

$$\left(-\frac{5}{3}\right)^2 + 1 = \sec^2\theta$$

$$\frac{34}{9} = \sec^2\theta$$

In quadrant II, $\sec\theta$ is negative, so

$$\sec\theta = -\sqrt{\frac{34}{9}} = -\frac{\sqrt{34}}{3}$$

Example 1 FINDING TRIGONOMETRIC FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT (continued)

(b) $\sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$ Quotient identity

$$\cos\theta\tan\theta = \sin\theta$$

$$\frac{1}{\sec\theta}\tan\theta = \sin\theta$$ Reciprocal identity

from part (a)

$$\frac{1}{-\frac{\sqrt{34}}{3}}\left(-\frac{5}{3}\right) = \sin\theta$$

$$\frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34} = \sin\theta$$

Example 1 FINDING TRIGONOMETRIC FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT (continued)

(c) $\cot(-\theta)$

$$\cot(-\theta) = \frac{1}{\tan(-\theta)}$$ Reciprocal identity

$$\cot(-\theta) = \frac{1}{-\tan\theta}$$ Negative-angle identity

$$\cot(-\theta) = \frac{1}{-\left(-\frac{5}{3}\right)} = \frac{3}{5}$$

Caution

To avoid a common error, when taking the square root, be sure to choose the sign based on the quadrant of θ and the function being evaluated.

WRITING ONE TRIGONOMETRIC FUNCITON IN TERMS OF ANOTHER

Write cos *x* in terms of tan *x*.

Since sec *x* is related to both cos *x* and tan *x* by identities, start with $1+\tan^2 x = \sec^2 x$.

$$\frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x}$$ Take reciprocals.

$$\frac{1}{1+\tan^2 x} = \cos^2 x$$ Reciprocal identity

$$\pm\sqrt{\frac{1}{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \cos x$$ Take the square root of each side.

$$\frac{\pm\sqrt{1+\tan^2 x}}{1+\tan^2 x} = \cos x$$ The sign depends on the quadrant of *x*.

Example 3 REWRITING AN EXPRESSION IN TERMS OF SINE AND COSINE

Write $\dfrac{1+\cot^2\theta}{1-\csc^2\theta}$ in terms of sin *θ* and cos *θ*, and then simplify the expression so that no quotients appear.

$$\frac{1+\cot^2\theta}{1-\csc^2\theta}=\frac{1+\dfrac{\cos^2\theta}{\sin^2\theta}}{1-\dfrac{1}{\sin^2\theta}}$$

Quotient identities

$$=\frac{\left(1+\dfrac{\cos^2\theta}{\sin^2\theta}\right)\sin^2\theta}{\left(1-\dfrac{1}{\sin^2\theta}\right)\sin^2\theta}$$

Multiply numerator and denominator by the LCD.

Example 3 REWRITING AN EXPRESSION IN TERMS OF SINE AND COSINE (cont'd)

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta - 1}$$

Distributive property

$$= \frac{1}{-\cos^2\theta}$$

Pythagorean identities

$$= -\sec^2\theta$$

Reciprocal identity

Caution

When working with trigonometric expressions and identities, be sure to write the argument of the function.

For example, we would *not* write $\sin^2 + \cos^2 = 1$. An argument such as θ is necessary in this identity.