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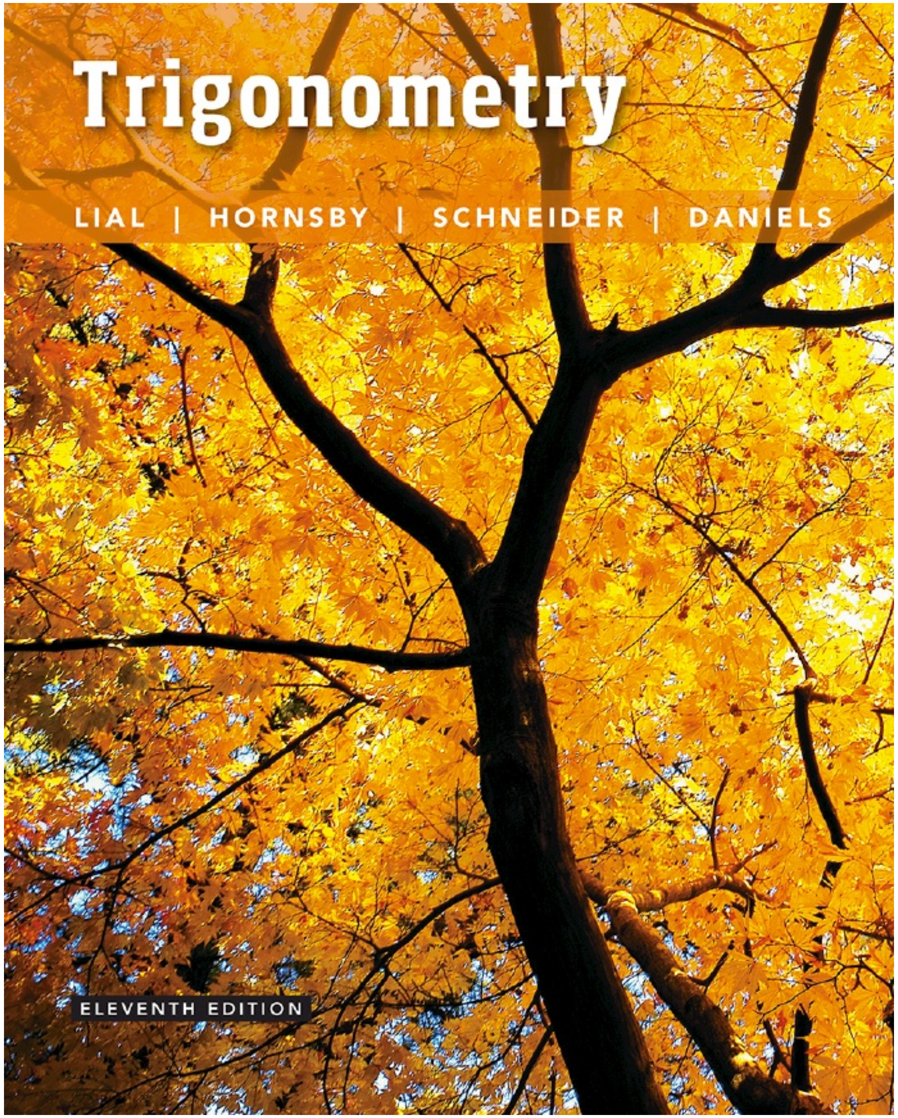
Trigonometric Identities



5.2 Verifying Trigonometric Identities

Strategies ▪ Verifying Identities by Working with One Side ▪ Verifying Identities by Working with Both Sides

Hints for Verifying Identities

- ***Learn the fundamental identities.***

Whenever you see either side of a fundamental identity, the other side should come to mind. ***Also, be aware of equivalent forms of the fundamental identities.***

- ***Try to rewrite the more complicated side*** of the equation so that it is identical to the simpler side.

Hints for Verifying Identities

- ***It is sometimes helpful to express all trigonometric functions in the equation in terms of sine and cosine*** and then simplify the result.

- ***Usually, any factoring or indicated algebraic operations should be performed.***

  For example, the expression

  $$\sin^2 x + 2\sin x + 1$$

  can be factored as $(\sin x + 1)^2$.

Hints for Verifying Identities

The sum or difference of two trigonometric expressions can be found in the same way as any other rational expression. For example,

$$\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=\frac{1\bullet\cos\theta}{\sin\theta\cos\theta}+\frac{1\bullet\sin\theta}{\cos\theta\sin\theta}$$
$$=\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}.$$

Hints for Verifying Identities

- ***As you select substitutions, keep in mind the side you are not changing, because it represents your goal.***

For example, to verify the identity

$$\tan^2 x + 1 = \frac{1}{\cos^2 x}$$

find an identity that relates tan *x* to cos *x*.

Since $\sec x = \frac{1}{\cos\theta}$ and $\sec^2 x = \tan^2 x + 1$ the secant function is the best link between the two sides.

Hints for Verifying Identities

- If an expression contains $1 + \sin x$, ***multiplying both numerator and denominator*** by $1 - \sin x$ would give $1 - \sin^2 x$, which could be replaced with $\cos^2 x$.

Similar procedures apply for $1 - \sin x$, $1 + \cos x$, and $1 - \cos x$.

Caution

The procedure for verifying identities is not the same as that of solving equations.

Techniques used in solving equations, such as adding the same term to each side, and multiplying each side bythe same term, should not be used when working with identities.

Verifying Identities by Working with One Side

To avoid the temptation to use algebraic properties of equations to verify identities, ***one strategy is to work with only one side and rewrite it to match the other side.***

Example 1 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that the following equation is an identity.

$$\cot\theta + 1 = \csc\theta(\cos\theta + \sin\theta)$$

Work with the right side since it is more complicated.

$$\begin{aligned}
\overbrace{\csc\theta(\cos\theta+\sin\theta)}^{\text{Right side of given equation}} &= \frac{1}{\sin\theta}(\cos\theta+\sin\theta) && \csc\theta = \frac{1}{\sin\theta} \\
&= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\sin\theta} && \text{Distributive property} \\
&= \underbrace{\cot\theta + 1}_{\text{Left side of given equation}} && \cot\theta = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that the following equation is an identity.

$$\tan^2 x\left(1+\cot^2 x\right)=\frac{1}{1-\sin^2 x}$$

$$\tan^2 x\left(1+\cot^2 x\right)=\tan^2 x+\tan^2 x\cot^2 x$$

Distributive property

Left side

$$=\tan^2 x+\tan^2 x\left(\frac{1}{\tan^2 x}\right)$$

$\cot x=\dfrac{1}{\tan x}$

$$=\tan^2 x+1$$

$$=\sec^2 x$$

$\tan^2 x+1=\sec^2 x$

$$=\frac{1}{\cos^2 x}$$

$\sec x=\dfrac{1}{\cos x}$

$$=\frac{1}{1-\sin^2 x}$$

Right side

$\cos^2 x=1-\sin^2 x$

Example 3 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that $\dfrac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t$ is an identity.

$$\frac{\tan t - \cot t}{\sin t \cos t} = \frac{\tan t}{\sin t \cos t} - \frac{\cot t}{\sin t \cos t} \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

$$= \tan t \cdot \frac{1}{\sin t \cos t} - \cot t \cdot \frac{1}{\sin t \cos t}$$

$$= \frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t}$$

$$= \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t} \qquad \tan t = \frac{\sin t}{\cos t};\ \cot t = \frac{\cos t}{\sin t}$$

$$= \sec^2 t - \csc^2 t \qquad \sec t = \frac{1}{\cos t};\ \csc t = \frac{1}{\sin t}$$

Example 4 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\sin x}{\cos x}$ is an identity.

$$\frac{1+\sin x}{\cos x}=\frac{(1+\sin x)(1-\sin x)}{\cos x(1-\sin x)}$$

Multiply by 1 in the form $\dfrac{1-\sin x}{1-\sin x}$

$$=\frac{1-\sin^2 x}{\cos x(1-\sin x)}$$

$$=\frac{\cos^2 x}{\cos x(1-\sin x)}$$

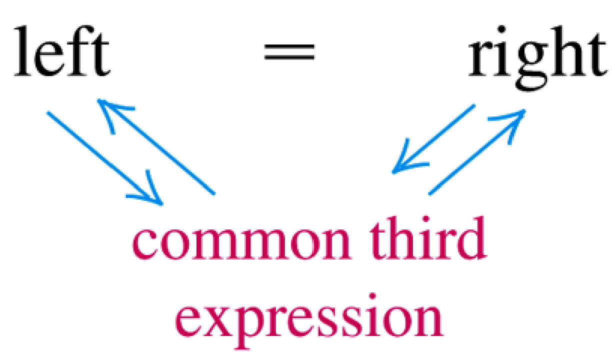
$\cos^2 x=1-\sin^2 x$

$$=\frac{\cos x}{1-\sin x}$$

Verifying Identities by Working with Both Sides

If both sides of an identity appear to be equally complex, the identity can be verified by working independently on each side until they are changed into a common third result.

Each step, on each side, must be reversible.



Example 5 VERIFYING AN IDENTITY (WORKING WITH BOTH SIDES)

Verify that $\dfrac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}=\dfrac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}$ is an identity.

Working with the left side:

$$\frac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}=\frac{(\sec\alpha+\tan\alpha)\cos\alpha}{(\sec\alpha-\tan\alpha)\cos\alpha}$$

Multiply by 1 in the form $\dfrac{\cos\alpha}{\cos\alpha}$

$$=\frac{\sec\alpha\cos\alpha+\tan\alpha\cos\alpha}{\sec\alpha\cos\alpha-\tan\alpha\cos\alpha}$$

Distributive property

$$=\frac{1+\dfrac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha}$$

$\sec\alpha\cos\alpha=1$

$\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}$

$$=\frac{1+\sin\alpha}{1-\sin\alpha}$$

Example 5 VERIFYING AN IDENTITY (WORKING WITH BOTH SIDES) (continued)

Working with the right side:

$$\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}=\frac{(1+\sin\alpha)^2}{\cos^2\alpha}$$

Factor the numerator.

$$=\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}$$

$\cos^2\alpha=1-\sin^2\alpha$

$$=\frac{(1+\sin\alpha)^2}{(1+\sin\alpha)(1-\sin\alpha)}$$

Factor the denominator.

$$=\frac{1+\sin\alpha}{1-\sin\alpha}$$

Example 5 VERIFYING AN IDENTITY (WORKING WITH BOTH SIDES) (continued)

$$\overbrace{\frac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}}^{\text{Left side of given equation}}=\underbrace{\frac{1+\sin\alpha}{1-\sin\alpha}}_{\text{Common third expression}}=\overbrace{\frac{1+2\sin\alpha+\sin^2\alpha}{\cos^2\alpha}}^{\text{Right side of given equation}}$$

So, the identity is verified.

Example 6 APPLYING A PYTHAGOREAN IDENTITY TO ELECTRONICS

Tuners in radios select a radio station by adjusting the frequency. A tuner may contain an inductor *L* and a capacitor, *C*. The energy stored in the inductor at time *t* is given by

$$L(t) = k\sin^2(2\pi Ft)$$

and the energy in the capacitor is given by

$$C(t) = k\cos^2(2\pi Ft)$$

where *F* is the frequency of the radio station and *k* is a constant.

Example 6 APPLYING A PYTHAGOREAN IDENTITY TO RADIOS (continued)

The total energy *E* in the circuit is given by

$$E(t) = L(t) + C(t)$$

Show that *E* is a constant function.*

An Inductor and a Capacitor

*(*Source*: Weidner, R. and R. Sells, *Elementary Classical Physics*, Vol. 2, Allyn & Bacon.)

Example 6 APPLYING A PYTHAGOREAN IDENTITY TO RADIOS (continued)

$$\begin{aligned} E(t) &= L(t) + C(t) \\ &= k\sin^2(2\pi Ft) + k\cos^2(2\pi Ft) \\ &= k\left[\sin^2(2\pi Ft) + \cos^2(2\pi Ft)\right] && \text{Factor.} \\ &= k \cdot 1 && \sin^2\theta + \cos^2\theta = 1 \\ & && (\theta = 2\pi Ft) \\ &= k \end{aligned}$$

Since *k* is a constant, *E*(*t*) is a constant function.