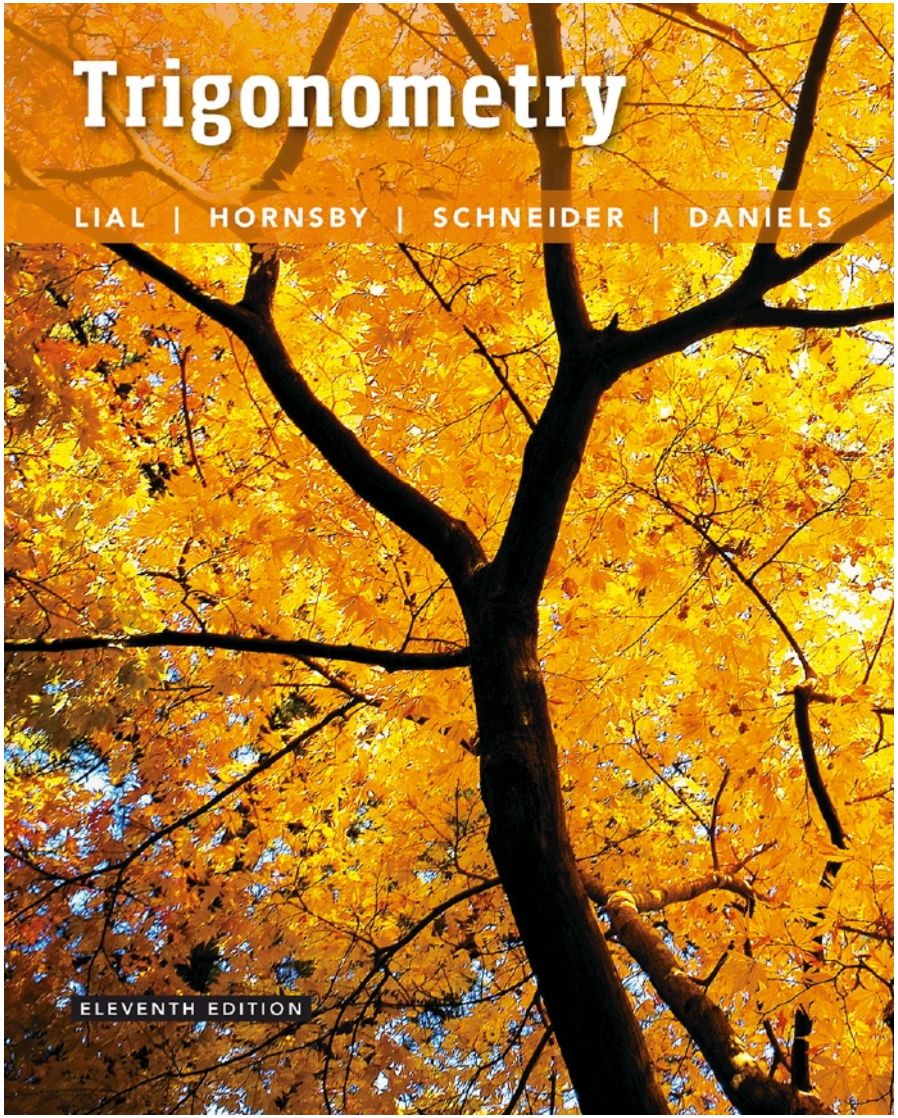
5

Trigonometric Identities

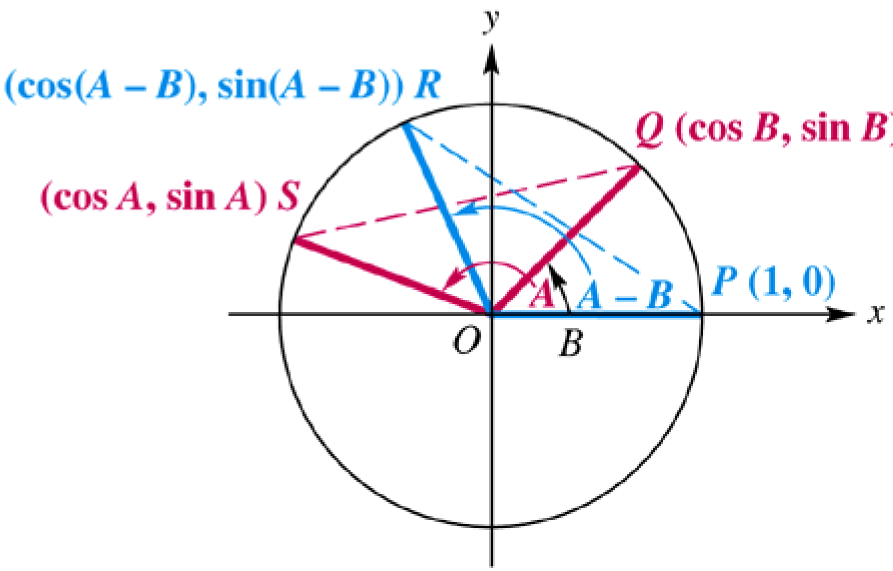


5.3 Sum and Difference Identities

Difference Identity for Cosine ▪ Sum Identity for Cosine ▪ Cofunction Identities ▪ Applications of the Sum and Difference Identities ▪ Verifying an Identity

Difference Identity for Cosine

Point Q is on the unit circle, so the coordinates of Q are (cos B, sin B).



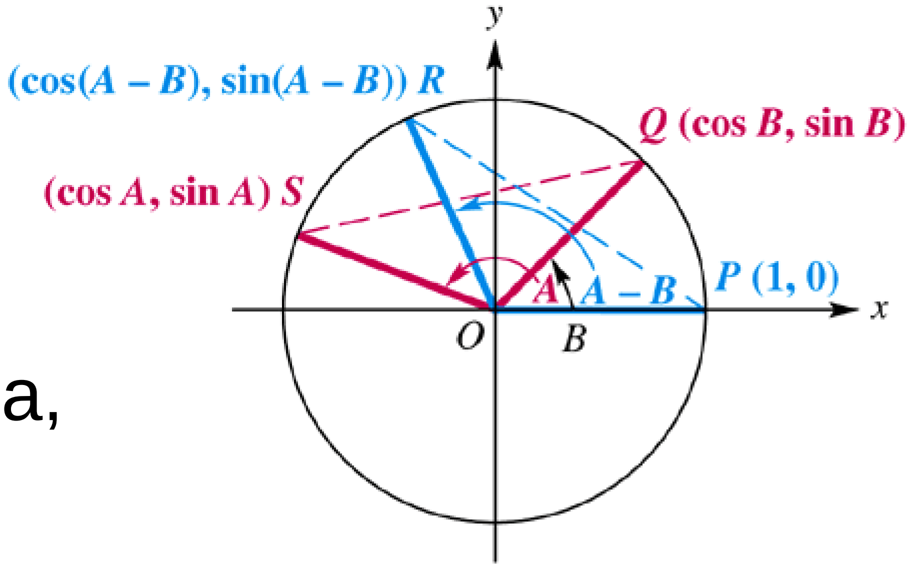
The coordinates of S are (cos A, sin A).

The coordinates of R are (cos(A – B), sin (A – B)).

$$m\angle SOQ = A - B$$

Difference Identity for Cosine

Since the central angles *SOQ* and *POR* are equal, *PR* = *SQ*.



Using the distance formula, since *PR* = *SQ*,

$$\sqrt{\left[\cos(A-B)-1\right]^2+\left[\sin(A-B)-0\right]^2}$$

$$=\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}$$

Difference Identity for Cosine

Square each side and clear parentheses:

$$\cos^2(A-B)-2\cos(A-B)+1+\sin^2(A-B)$$
$$=\cos^2 A-2\cos A\cos B+\cos^2 B$$
$$+\sin^2 A-2\sin A\sin B+\sin^2 B$$

$\sin^2\theta+\cos^2\theta=1$ for any value of θ.

$$2-2\cos(A-B)=2-2\sin A\sin B-2\cos A\cos B$$

Subtract 2 and divide by –2:

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

Sum Identity for Cosine

To find a similar expression for cos(*A* + *B*) rewrite *A* + *B* as *A* – (–*B*) and use the identity for cos(*A* – *B*).

$$\cos(A+B) = \cos[A-(-B)]$$

$$= \cos A \cos(-B) + \sin A \sin(-B)$$

Cosine difference identity

$$= \cos A \cos B + \sin A(-\sin B)$$

Negative-angle identities

$$= \cos A \cos B - \sin A \sin B$$

Cosine of a Sum or Difference

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Example 1(a) FINDING EXACT COSINE FUNCTION VALUES

Find the *exact* value of cos 15°.

$$\cos 15^\circ = \cos\left(45^\circ - 30^\circ\right)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \cos 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{2}$$

Find the *exact* value of $\cos \frac{5\pi}{12}$.

$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

$$= \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 1(c) FINDING EXACT COSINE FUNCTION VALUES

Find the *exact* value of cos 87°cos 93° – sin 87°sin 93°.

$$\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ = \cos(87^\circ + 93^\circ)$$
$$= \cos(180^\circ)$$
$$= -1$$

Cofunction Identities

$$\cos(90° - \theta) = \sin\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sin(90° - \theta) = \cos\theta \qquad \sec(90° - \theta) = \csc\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \csc(90° - \theta) = \sec\theta$$

The same identities can be obtained for a real number domain by replacing 90° with $\frac{\pi}{2}$.

Example 2 USING COFUNCTION IDENTITIES TO FIND θ

Find one value of θ or x that satisfies each of the following.

(a) $\cot\theta = \tan 25°$

$$\cot\theta = 25°$$
$$\tan(90° - \theta) = \tan 25°$$
$$90° - \theta = 25°$$
$$\theta = 65°$$

(b) $\sin\theta = \cos(–30°)$

$$\sin\theta = \cos(-30°)$$
$$\cos(90° - \theta) = \cos(-30°)$$
$$90° - \theta = -30°$$
$$\theta = 120°$$

Example 2 USING COFUNCTION IDENTITIES TO FIND θ (continued)

Find one value of θ or x that satisfies the following.

(c) $\csc \frac{3\pi}{4} = \sec x$

$$\csc \frac{3\pi}{4} = \sec x$$

$$\csc \frac{3\pi}{4} = \csc\left(\frac{\pi}{2} - x\right)$$

$$\frac{3\pi}{4} = \frac{\pi}{2} - x$$

$$x = -\frac{\pi}{4}$$

Note

Because trigonometric (circular) functions are periodic, the solutions in **Example 2** are not unique. We give only one of infinitely many possibilities.

Applying the Sum and Difference Identities

If either angle A or B in the identities for $\cos(A + B)$ and $\cos(A - B)$ is a quadrantal angle, then the identity allows us to write the expression in terms of a single function of A or B.

Example 3 REDUCING cos (*A* – *B*) TO A FUNCTION OF A SINGLE VARIABLE

Write cos(180° – *θ*) as a trigonometric function of *θ* alone.

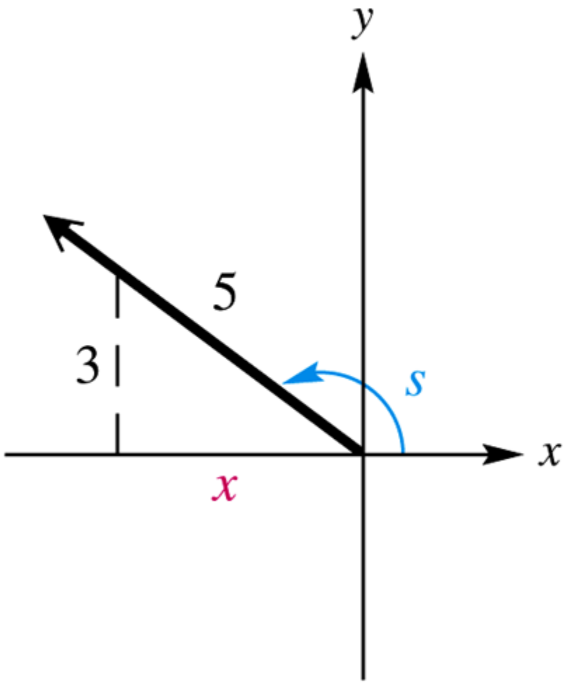
$$\begin{aligned}\cos(180^\circ - \theta) &= \cos 180^\circ \cos\theta + \sin 180^\circ \sin\theta \\ &= (-1)\cos\theta + (0)\sin\theta \\ &= -\cos\theta\end{aligned}$$

Example 4 FINDING cos (*s* + *t*) GIVEN INFORMATION ABOUT *s* AND *t*

Suppose that $\sin s = \frac{3}{5}$, $\cos t = -\frac{12}{13}$, and both *s* and *t* are in quadrant II. Find cos(*s* + *t*).

Method 1

Sketch an angle *s* in quadrant II such that $\sin s = \frac{3}{5}$. Since $\sin s = \frac{3}{5} = \frac{y}{r}$, let *y* = 3 and *r* = 5.



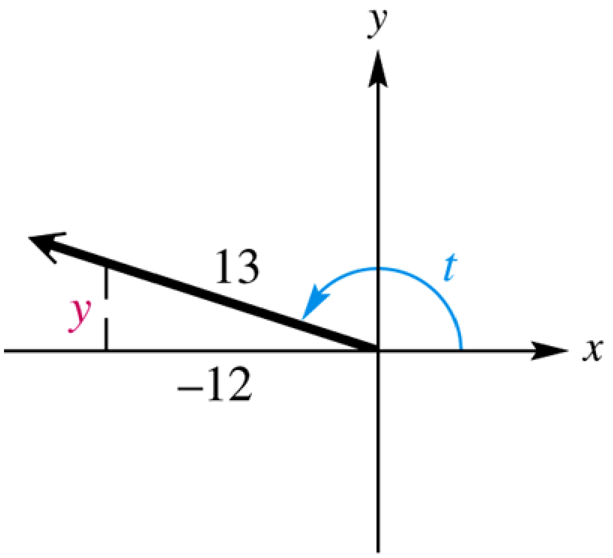
The Pythagorean theorem gives

$$x^2 + 3^2 = 5^2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$

Since *s* is in quadrant II, *x* = –4 and $\cos s = -\frac{4}{5}$.

Example 4 FINDING cos (*s* + *t*) GIVEN INFORMATION ABOUT *s* AND *t* (cont.)

Sketch an angle *t* in quadrant II such that $\cos t = -\frac{12}{13}$. Since $\cos t = -\frac{12}{13} = \frac{x}{r}$, let $x = -12$ and $r = 13$.



The Pythagorean theorem gives

$$(-12)^2 + y^2 = 13^2 \Rightarrow y^2 = 25 \Rightarrow y = \pm 5$$

Since *t* is in quadrant II, $y = 5$ and $\sin t = \frac{5}{13}$.

Example 4 FINDING cos (*s* + *t*) GIVEN INFORMATION ABOUT *s* AND *t* (cont.)

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

$$= -\frac{4}{5}\left(-\frac{12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$$

$$= \frac{48}{65} - \frac{15}{65}$$

$$= \frac{33}{65}$$

Method 2

We use Pythagorean identities here. To find cos *s*, recall that $\sin^2 s + \cos^2 s = 1$, where *s* is in quadrant II.

$$\left(\frac{3}{5}\right)^2 + \cos^2 s = 1$$

sin *s* = 3/5

$$\frac{9}{25} + \cos^2 s = 1$$

Square.

$$\cos^2 s = \frac{16}{25}$$

Subtract 9/25

$$\cos s = -\frac{4}{5}$$

cos *s* < 0 because *s* is in quadrant II.

Example 4 FINDING cos (*s* + *t*) GIVEN INFORMATION ABOUT *s* AND *t* (cont.)

To find sin *t*, we use $\sin^2 t + \cos^2 t = 1$, where *t* is in quadrant II.

$$\sin^2 t + \left(-\frac{12}{13}\right)^2 = 1 \qquad \cos t = -12/13$$

$$\sin^2 t + \frac{144}{169} = 1 \qquad \text{Square.}$$

$$\sin^2 t = \frac{25}{169} \qquad \text{Subtract } 144/169$$

$$\sin t = \frac{5}{13} \qquad \sin t > 0 \text{ because } t \text{ is in quadrant II.}$$

From this point, the problem is solved using

$\cos(s+t) = \cos s \cos t - \sin s \sin t$ (see Method 1).

Example 5 APPLYING THE COSINE DIFFERENCE IDENTITY TO VOLTAGE

Common household electric current is called **alternating current** because the current alternates direction within the wires. The voltage *V* in a typical 115-volt outlet can be expressed by the function

$$V(t) = 163 \sin \omega t,$$

where ω is the angular speed (in radians per second) of the rotating generator at the electrical plant, and *t* is time measured in seconds. (*Source*: Bell, D., *Fundamentals of Electric Circuits*, Fourth Edition, Prentice-Hall, 1988.)

Example 5 APPLYING THE COSINE DIFFERENCE IDENTITY TO VOLTAGE (continued)

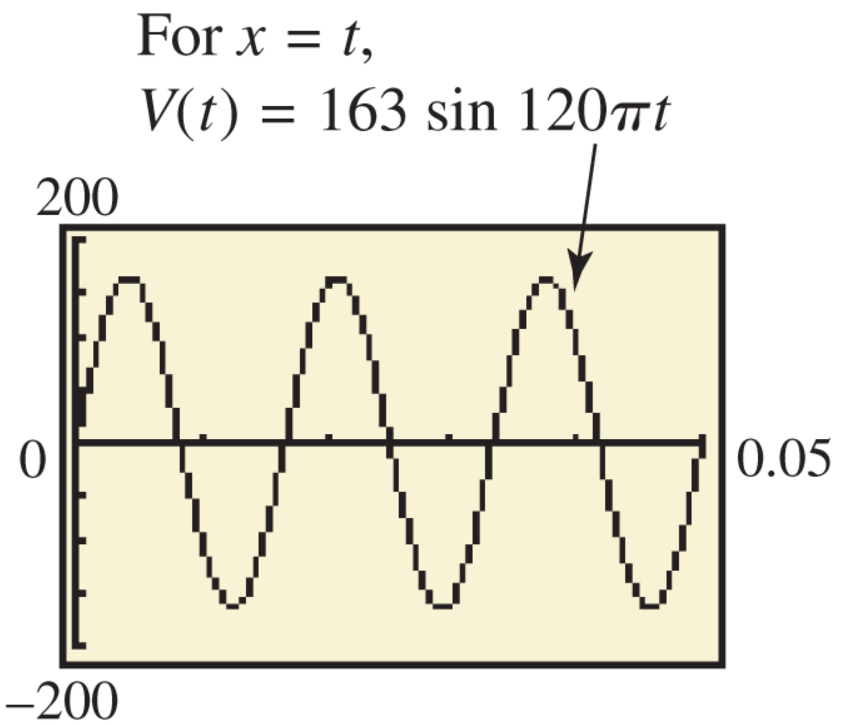
(a) It is essential for electric generators to rotate at precisely 60 cycles per sec so household appliances and computers will function properly. Determine ω for these electric generators.

Each cycle is 2π radians at 60 cycles per sec, so the angular speed is $\omega = 60(2\pi) = 120\pi$ radians per sec.

Example 5 APPLYING THE COSINE DIFFERENCE IDENTITY TO VOLTAGE (continued)

(b) Graph V in the window [0, 0.05] by [–200, 200].

$$V(t) = 163 \sin \omega t = 163 \sin 120\pi t$$



(c) Determine a value of ϕ so that the graph of $V(t) = 163\cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163\sin\omega t$.

Using the negative-angle identity for cosine and a cofunction identity gives

$$\cos\left(x - \frac{\pi}{2}\right) = \cos\left[-\left(\frac{\pi}{2} - x\right)\right] = \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

Therefore, if $\phi = \frac{\pi}{2}$,

$$V(t) = 163\cos(\omega t - \phi) = 163\cos\left(\omega t - \frac{\pi}{2}\right) = 163\sin\omega t.$$

Example 6 VERIFYING AN IDENTITY

Verify that the following equation is an identity.

$$\sec\left(\frac{3\pi}{2}-x\right)=-\csc x$$

Work with the more complicated left side.

$$\sec\left(\frac{3\pi}{2}-x\right)=\frac{1}{\cos\left(\frac{3\pi}{2}-x\right)}$$

$$=\frac{1}{\cos\frac{3\pi}{2}\cos x+\sin\frac{3\pi}{2}\sin x}$$

Example 6 VERIFYING AN IDENTITY (continued)

$$\frac{1}{\cos\frac{3\pi}{2}\cos x+\sin\frac{3\pi}{2}\sin x}=\frac{1}{0\bullet\cos x+(-1)\sin x}$$

$$=\frac{1}{-\sin x}$$

$$=-\csc x$$

The left side is identical to the right side, so the given equation is an identity.