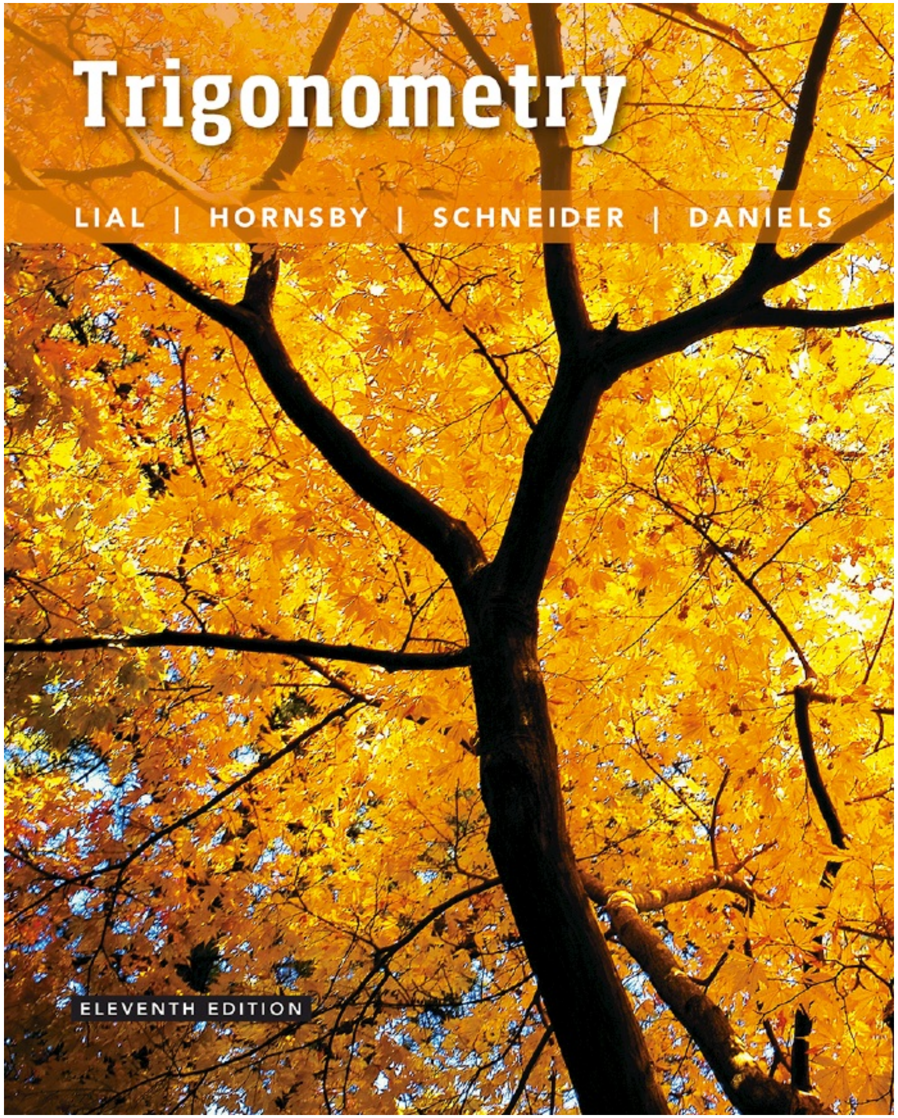
5

Trigonometric Identities



5.4 Sum and Difference Identities for Sine and Tangent

Sum and Difference Identities for Sine ▪ Sum and Difference Identities for Tangent ▪ Applying the Sum and Difference Identities ▪ Verifying an Identity

Sum and Difference Identities for Sine

We can use the cosine sum and difference identities to derive similar identities for sine and tangent.

$$\sin(A+B) = \cos\left[90^\circ - (A+B)\right]$$ Cofunction identity

$$= \cos\left[(90^\circ - A) - B\right]$$

$$= \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B$$

Cosine difference identity

$$= \sin A \cos B + \cos A \sin B$$

Cofunction identities

Sum and Difference Identities for Sine

$$\sin(A - B) = \sin[A + (-B)]$$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

Sine sum identity

$$= \sin A \cos B - \cos A \sin B$$

Negative-angle identities

Sine of a Sum or Difference

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Sum and Difference Identities for Tangent

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

Fundamental identity

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Sum identities

$$= \frac{\dfrac{\sin A \cos B + \cos A \sin B}{1}}{\dfrac{\cos A \cos B - \sin A \sin B}{1}} \cdot \frac{\dfrac{1}{\cos A \cos B}}{\dfrac{1}{\cos A \cos B}}$$

Multiply numerator and denominator by 1.

Sum and Difference Identities for Tangent

$$\tan(A+B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

Multiply.

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

Simplify.

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Fundamental identity

Replace *B* with –*B* and use the fact that tan(–*B*) = –tan *B* to obtain the identity for the tangent of the difference of two angles.

Tangent of a Sum or Difference

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Example 1(a) FINDING EXACT SINE AND TANGENT FUNCTION VALUES

Find the *exact* value of sin 75°.

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 1(b) FINDING EXACT SINE AND TANGENT FUNCTION VALUES

Find the *exact* value of $\tan\frac{7\pi}{12}$.

$$\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1-3} = \frac{4+2\sqrt{3}}{-2}$$

$$= -2-\sqrt{3}$$

Example 1(c) FINDING EXACT SINE AND TANGENT FUNCTION VALUES

Find the *exact* value of $\sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ$.

$$\begin{aligned}\sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ &= \sin(40^\circ - 160^\circ)\\ &= \sin(-120^\circ)\\ &= -\sin 120^\circ\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Example 2 WRITING FUNCTIONS AS EXPRESSIONS INVOLVING FUNCTIONS OF θ

Write each function as an expression involving functions of θ.

(a) $$=\cos(30^\circ+\theta)=\cos 30^\circ\cos\theta-\sin 30^\circ\sin\theta$$

$$=\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta=\frac{\sqrt{3}\cos\theta-\sin\theta}{2}$$

(b) $$\tan(45^\circ-\theta)=\frac{\tan 45^\circ-\tan\theta}{1+\tan 45^\circ\tan\theta}=\frac{1-\tan\theta}{1+\tan\theta}$$

(c) $$\sin(180^\circ-\theta)=\sin 180^\circ\cos\theta-\cos 180^\circ\sin\theta$$

$$=0\bullet\cos\theta-(-1)\sin\theta$$

$$=\sin\theta$$

Example 3 FINDING FUNCTION VALUES AND THE QUADRANT OF *A* + *B*

Suppose that *A* and *B* are angles in standard position with $\sin A = \frac{4}{5}, \frac{\pi}{2} < A < \pi,$ and $\cos B = -\frac{5}{13}, \pi < B < \frac{3\pi}{2}$. Find each of the following.

(a) $\sin(A + B)$

(b) $\tan(A + B)$

(c) the quadrant of $A + B$

Example 3 FINDING FUNCTION VALUES AND THE QUADRANT OF *A* + *B* (continued)

The identity for sin(*A* + *B*) involves sin *A*, cos *A*, sin *B*, and cos *B.* The identity for tan(*A* + *B*) requires tan *A* and tan *B*. We must find cos *A*, tan *A*, sin *B* and tan *B*.

Because *A* is in quadrant II, cos *A* is negative and tan *A* is negative.

$$\sin^2 A + \cos^2 A = 1$$

$$\left(\frac{4}{5}\right)^2 + \cos^2 A = 1$$

$$\cos^2 A = \frac{9}{25} \Rightarrow \cos A = -\frac{3}{5}$$

Example 3 FINDING FUNCTION VALUES AND THE QUADRANT OF *A* + *B* (continued)

Because *B* is in quadrant III, sin *B* is negative and tan *B* is positive.

$$\sin^2 B + \cos^2 B = 1$$

$$\sin^2 B + \left(-\frac{5}{13}\right)^2 = 1$$

$$\sin^2 B = \frac{144}{169} \quad \Rightarrow \sin B = -\frac{12}{13}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3} \qquad \tan B = \frac{\sin B}{\cos B} = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

FINDING FUNCTION VALUES AND THE QUADRANT OF *A* + *B* (continued)

(a) $$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$=\frac{4}{5}\left(-\frac{5}{13}\right)+\left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$=-\frac{20}{65}+\frac{36}{65}=\frac{16}{65}$$

(b) $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$=\frac{-\frac{4}{3}+\frac{12}{5}}{1-\left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)}=\frac{\frac{16}{15}}{\frac{63}{15}}=\frac{16}{63}$$

Example 3 FINDING FUNCTION VALUES AND THE QUADRANT OF *A* + *B* (continued)

(c) From parts (a) and (b), sin $(A + B) > 0$ and tan $(A + B) > 0$.

The only quadrant in which the values of both the sine and the tangent are positive is quadrant I, so $(A + B)$ is in quadrant I.

Example 4 VERIFYING AN IDENTITY USING SUM AND DIFFERENCE IDENTITIES

Verify that the equation is an identity.

$$\sin\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{3}+\theta\right)=\cos\theta$$

$$\sin\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{3}+\theta\right)$$

$$=\left(\sin\frac{\pi}{6}\cos\theta+\sin\theta\cos\frac{\pi}{6}\right)+\left(\cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta\right)$$

$$=\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)+\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)$$

$$=\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta=\cos\theta$$