Ex. 2 Given that $\cos(2\theta) = \frac{4}{5}$ and $90° < \theta < 180°$

Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$.

Sol: $\cos(2\theta) = 1 - 2\sin^2\theta$

$$\frac{4}{5} = 1 - 2\sin^2\theta \rightarrow -\frac{1}{5} = -2\sin^2\theta$$

$$\rightarrow \frac{1}{10} = \sin^2\theta \rightarrow \sin\theta = \pm\frac{1}{\sqrt{10}} = \pm\frac{\sqrt{10}}{10}$$

Since θ is in quadrant II, $\boxed{\sin\theta = \frac{\sqrt{10}}{10}}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{1}{10} + \cos^2\theta = 1 \rightarrow \cos^2\theta = \frac{9}{10}$$

$$\rightarrow \cos\theta = \pm\sqrt{\frac{9}{10}} = \pm\frac{3}{\sqrt{10}} = \pm\frac{3\sqrt{10}}{10}$$

Since θ is in quadrant II, $\boxed{\cos\theta = -\frac{3\sqrt{10}}{10}}$

$\rightarrow$ We can easily find the rest.

Ex. 3 Verify the identity

$$\cot(\theta)\cdot\sin(2\theta) = 1 + \cos(2\theta)$$

$$\text{LHS} = \frac{\cos(\theta)}{\cancel{\sin(\theta)}}\cdot 2\cdot\cancel{\sin(\theta)}\cdot\cos(\theta)$$

$$= \cos(\theta)\cdot 2\cdot\cos(\theta) = 2\cos(\theta)\cos(\theta)$$

$$= 2\cos^2(\theta)$$

$$\text{RHS} = \cancel{1} + 2\cos^2(\theta) \cancel{-1} = 2\cos^2(\theta)$$

So, LHS = RHS and we are done!

Ex. 4. Find the exact value of the expression

$$\sin 15^\circ \cos 15^\circ$$

$$?$$

$$\underbrace{\sin(30^\circ)}_{} = \sin(2\cdot 15^\circ) = 2\boxed{\sin 15^\circ \cos 15^\circ}$$

$$?$$

$$\frac{1}{2} = 2\cdot\boxed{\sin 15^\circ\cdot\cos 15^\circ}$$

So, $\boxed{\sin 15^\circ \cos 15^\circ = \frac{1}{4}.}$

Ex. 5. Find the exact value of

$$\cos^2(22.5^\circ) - \sin^2(22.5^\circ)$$

By the double angle identity for cosine

$$\cos^2(22.5^\circ) - \sin^2(22.5) = \cos(45^\circ) = \frac{\sqrt{2}}{2}.$$

Ex. 6. Write $\sin(3x)$ in terms of $\sin(x)$

Sum Identity for Sine

$$\sin(3x) = \sin(2x + x) = \sin(2x)\cdot\cos(x) + \cos(2x)\cdot\sin(x)$$

$$= (2\sin x\cos x)\cdot\cos x + (1 - 2\sin^2 x)\cdot\sin x$$

$$= 2\sin x\cdot\cos^2 x + \sin x - 2\sin^3 x$$

$$= 2\sin x\cdot(1 - \sin^2 x) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$\boxed{\sin(3x) = 3\sin x - 4\sin^3 x}$$

Obj 2: Product-to-Sum and Sum-to-Product Identities

$$\cos(A+B) = \cos A\cos B - \cancel{\sin A \sin B}$$
$$\cos(A-B) = \cos A\cos B + \cancel{\sin A \sin B}$$

+ (Add these equalities side-by-side)

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

Divide both sides by 2 and rewrite:

$$\underbrace{\cos A\cos B}_{\text{Product}} = \frac{1}{2}\Big[\underbrace{\cos(A+B) + \cos(A-B)}_{\text{Sum}}\Big]$$

product-to-sum

Ex. $\sin(A+B) = ?$

$\sin(A-B) = ?$

→ Derive another product to sum identity.

$$\underbrace{\sin A \cos B}_{\text{product}} = \frac{1}{2}\Big[\underbrace{\sin(A+B) + \sin(A-B)}_{\text{Sum}}\Big]$$

What happens if we subtract the original identities side-by-side?

$$\cos(A-B) = \cancel{\cos A \cos B} + \sin A \sin B$$
$$\cos(A+B) = \cancel{\cos A \cos B} - \sin A \sin B$$

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$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$$

$$\underbrace{\sin A \sin B}_{\text{Product}} = \frac{1}{2}\Big[\underbrace{\cos(A-B) - \cos(A+B)}_{\text{Sum}}\Big]$$

Last product-to-sum identity:

$$\underbrace{\cos A \sin B} = \frac{1}{2}\Big[\underbrace{\sin(A+B) - \sin(A-B)}\Big]$$