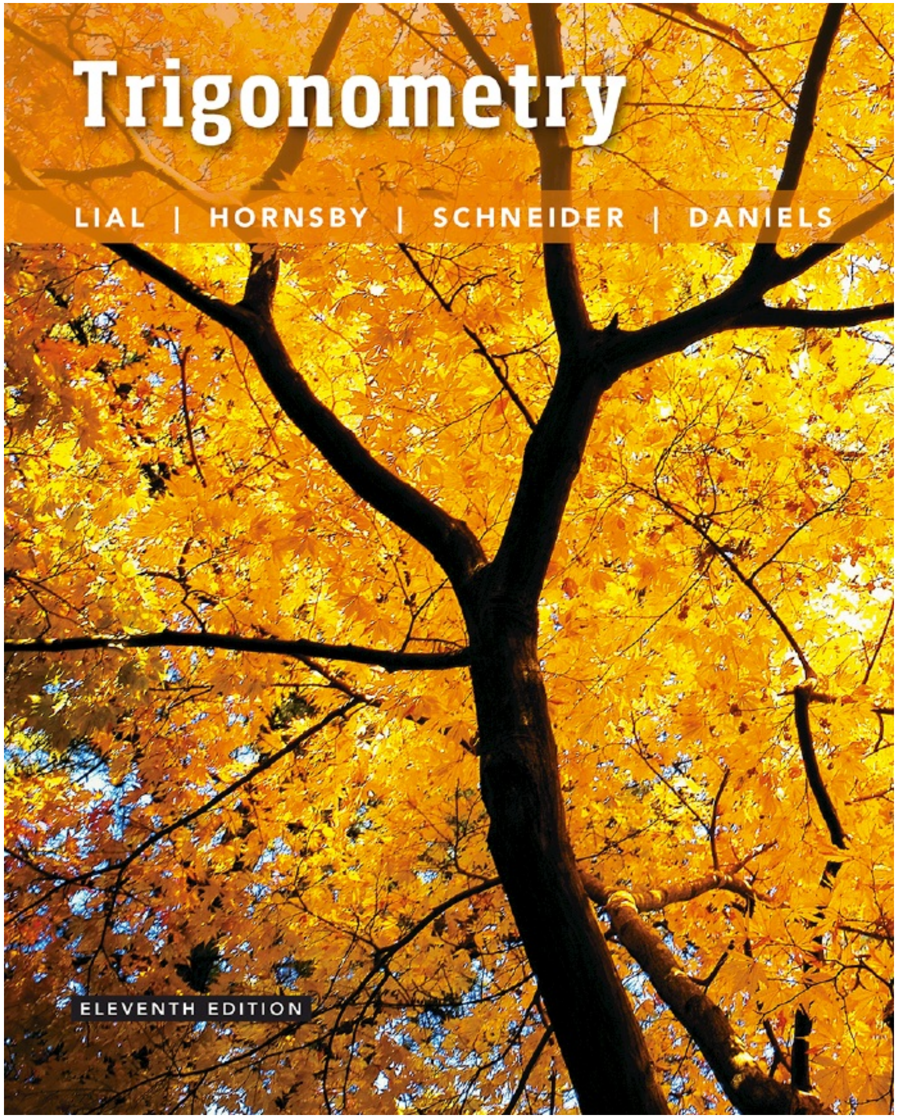
5

Trigonometric Identities



5.5 Double-Angle

Double-Angle Identities ▪ An Application ▪ Product-to-Sum and Sum-to-Product Identities

Double-Angle Identities

We can use the cosine sum identity to derive double-angle identities for cosine.

$$\cos 2A = \cos(A + A)$$

$$= \cos A \cos A - \sin A \sin A$$

Cosine sum identity

$$= \cos^2 A - \sin^2 A$$

Double-Angle Identities

There are two alternate forms of this identity.

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= \left(1 - \sin^2 A\right) - \sin^2 A \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - \left(1 - \cos^2 A\right) \\ &= 2\cos^2 A - 1\end{aligned}$$

Double-Angle Identities

We can use the sine sum identity to derive a double-angle identity for sine.

$$\sin 2A = \sin(A + A)$$

$$= \sin A \cos A + \cos A \sin A$$

Sine sum identity

$$= 2 \sin A \cos A$$

Double-Angle Identities

We can use the tangent sum identity to derive a double-angle identity for tangent.

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} \quad \text{Tangent sum identity}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

Double-Angle Identities

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \cos 2A = 1 - 2\sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1 \qquad \sin 2A = 2\sin A \cos A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Example 1 FINDING FUNCTION VALUES OF 2θ GIVEN INFORMATION ABOUT θ

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

To find $\sin 2\theta$, we must first find the value of $\sin\theta$.

$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1 \Rightarrow \sin^2\theta = \frac{16}{25} \Rightarrow \sin\theta = -\frac{4}{5}$$

Now use the double-angle identity for sine.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}$$

Now find $\cos 2\theta$, using the first double-angle identity for cosine (any of the three forms may be used).

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

Example 1 FINDING FUNCTION VALUES OF 2*θ* GIVEN INFORMATION ABOUT *θ* (cont.)

Now find tan *θ* and then use the tangent double-angle identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

Example 1 FINDING FUNCTION VALUES OF 2*θ* GIVEN INFORMATION ABOUT *θ* (cont.)

Alternatively, find tan 2*θ* by finding the quotient of sin 2*θ* and cos 2*θ*.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

Example 2 FINDING FUNCTION VALUES OF θ GIVEN INFORMATION ABOUT 2θ

Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ and $90° < \theta < 180°$.

We must obtain a trigonometric function value of θ alone.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

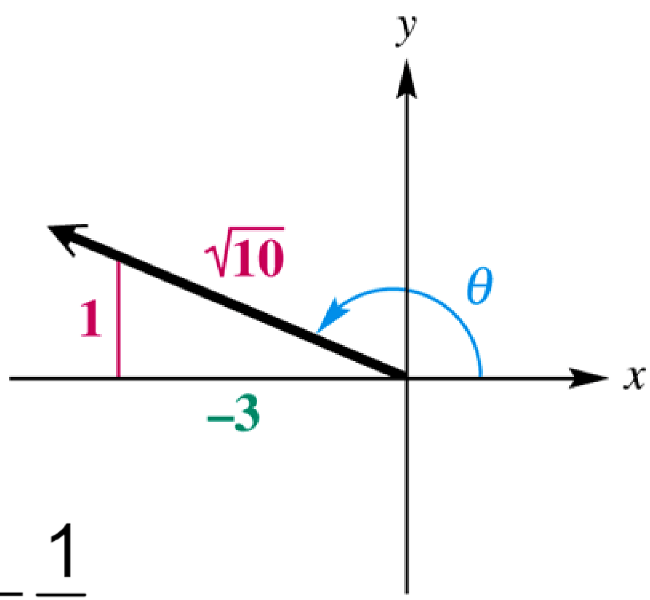
$$\frac{4}{5} = 1 - 2\sin^2\theta \Rightarrow \frac{1}{10} = \sin^2\theta$$

θ is in quadrant II, so sin θ is positive.

$$\sin\theta = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$$

Example 2 FINDING FUNCTION VALUES OF θ GIVEN INFORMATION ABOUT 2θ (cont.)

Use a right triangle in quadrant II to find the values of cos θ and tan θ.



$$r=\sqrt{10},\ y=1,\ x=-3$$

Use the Pythagorean theorem to find *x*.

$$\cos\theta=-\frac{3}{\sqrt{10}}=-\frac{3\sqrt{10}}{10}\qquad \tan\theta=-\frac{1}{3}$$

$$\cot\theta=-3\qquad \sec\theta=-\frac{\sqrt{10}}{3}\qquad \csc\theta=\sqrt{10}$$

Example 3 VERIFYING AN IDENTITY

Verify that $\cot x \sin 2x = 1 + \cos 2x$ is an identity.

$$\cot x \sin 2x = \frac{\cos x}{\sin x} \sin 2x$$

Quotient identity

$$= \frac{\cos x}{\sin x}(2 \sin x \cos x)$$

Double-angle identity

$$= 2\cos^2 x$$

$$= 1 + \cos 2x$$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow$
$2\cos^2 x = 1 + \cos 2x$

Example 4 SIMPLIFYING EXPRESSIONS USING DOUBLE-ANGLE IDENTITIES

Simplify each expression.

(a) $\cos^2 7x - \sin^2 7x = \cos 2(7x)$ $\quad$ $\cos 2A = \cos^2 A - \sin^2 A$

$$= \cos 14x$$

(b) $\sin 15^\circ \cos 15^\circ = \frac{1}{2} \cdot 2 \sin 15^\circ \cos 15^\circ$ $\quad$ Multiply by 1.

$$= \frac{1}{2}(2 \sin 15^\circ \cos 15^\circ)$$

$$= \frac{1}{2} \sin(2 \cdot 15^\circ)$$ $\quad$ $\sin 2A = 2 \sin A \cos A$

$$= \frac{1}{2} \sin 30^\circ = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

▶ Example 5 DERIVING A MULTIPLE-ANGLE IDENTITY

Write sin 3*x* in terms of sin *x*.

$$\sin 3x = \sin(2x + x)$$

$$= \sin 2x \cos x + \cos 2x \sin x \quad \text{Sine sum identity}$$

$$= (2\sin x \cos x)\cos x + \left(\cos^2 x - \sin^2 x\right)\sin x$$

Double-angle identities

$$= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x$$

$$= 2\sin x\left(1 - \sin^2 x\right) + \left(1 - \sin^2 x\right)\sin x - \sin^3 x$$

$\cos^2\theta = 1 - \sin^2\theta$

$$= 2\sin x - 2\sin^3 x + \sin x - \sin^3 x - \sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

Example 6 DETERMINING WATTAGE CONSUMPTION

If a toaster is plugged into a common household outlet, the wattage consumed is not constant. Instead, it varies at a high frequency according to the model

$$W = \frac{V^2}{R},$$

where V is the voltage and R is a constant that measures the resistance of the toaster in ohms.*

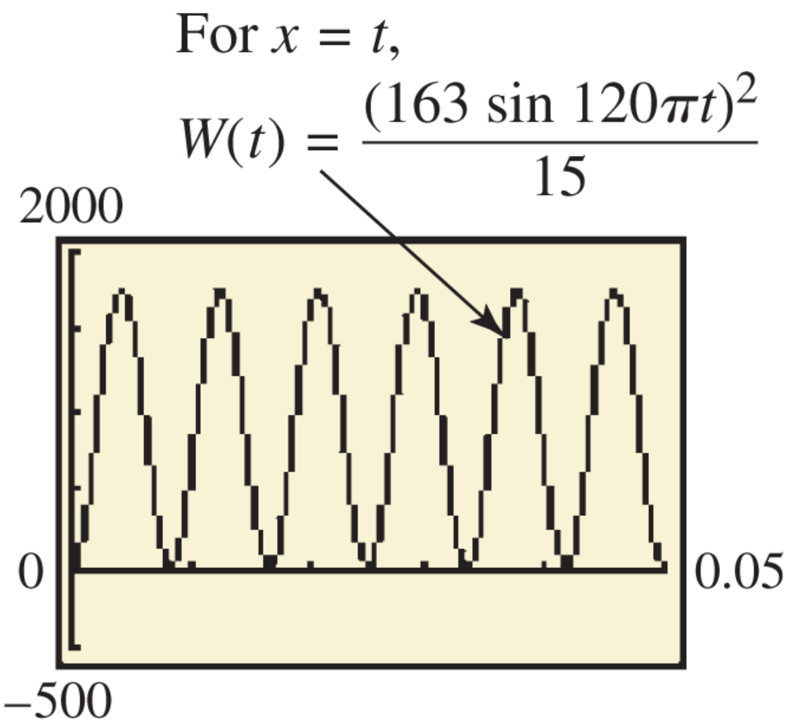
Graph the wattage W consumed by a typical toaster with $R = 15$ and $V = 163 \sin 120\pi t$ in the window [0, 0.05] by [–500, 2000]. How many oscillations are there?

***(*Source*: Bell, D., *Fundamentals of Electric Circuits*, Fourth Edition, Prentice-Hall.)**

Example 6 DETERMINING WATTAGE CONSUMPTION (continued)

Substituting the given values into the wattage equation gives

$$W = \frac{V^2}{R} = \frac{(163 \sin 120\pi t)^2}{15}.$$



The graph shows that there are six oscillations.

Product-to-Sum Identities

We can add the identities for cos(*A* + *B*) and cos(*A* – *B*) to derive a product-to-sum identity for cosines.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

Product-to-Sum Identities

Similarly, subtracting cos(*A* + *B*) from cos(*A* – *B*) gives a product-to-sum identity for sines.

$$
\begin{array}{r}
\cos(A-B)=\cos A\cos B+\sin A\sin B\\
\cos(A+B)=\cos A\cos B-\sin A\sin B\\
\hline
\cos(A-B)-\cos(A+B)=2\sin A\sin B\qquad\qquad\qquad\quad
\end{array}
$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Product-to-Sum Identities

Using the identities for sin(A + B) and sin(A – B) in the same way, we obtain two more identities.

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

Product-to-Sum Identities

$$\cos A\cos B=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

$$\sin A\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$

$$\cos A\sin B=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

Example 7 USING A PRODUCT-TO-SUM IDENTITY

Write 4 cos 75° sin 25° as the sum or difference of two functions.

$$4\cos 75^\circ \sin 25^\circ = 4 \cdot \frac{1}{2}\left[\left(\sin 75^\circ + 25^\circ\right) - \sin\left(75^\circ - 25^\circ\right)\right]$$

$$= 2\sin 100^\circ - 2\sin 50^\circ$$

Sum-to-Product Identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Example 8 USING A SUM-TO-PRODUCT IDENTITY

Write $\sin 2\theta - \sin 4\theta$ as a product of two functions.

$$\begin{aligned}\sin 2\theta - \sin 4\theta &= 2\cos\left(\frac{2\theta + 4\theta}{2}\right)\sin\left(\frac{2\theta - 4\theta}{2}\right)\\ &= 2\cos\left(\frac{6\theta}{2}\right)\sin\left(-\frac{2\theta}{2}\right)\\ &= 2\cos 3\theta \sin(-\theta)\\ &= -2\cos 3\theta \sin\theta\end{aligned}$$