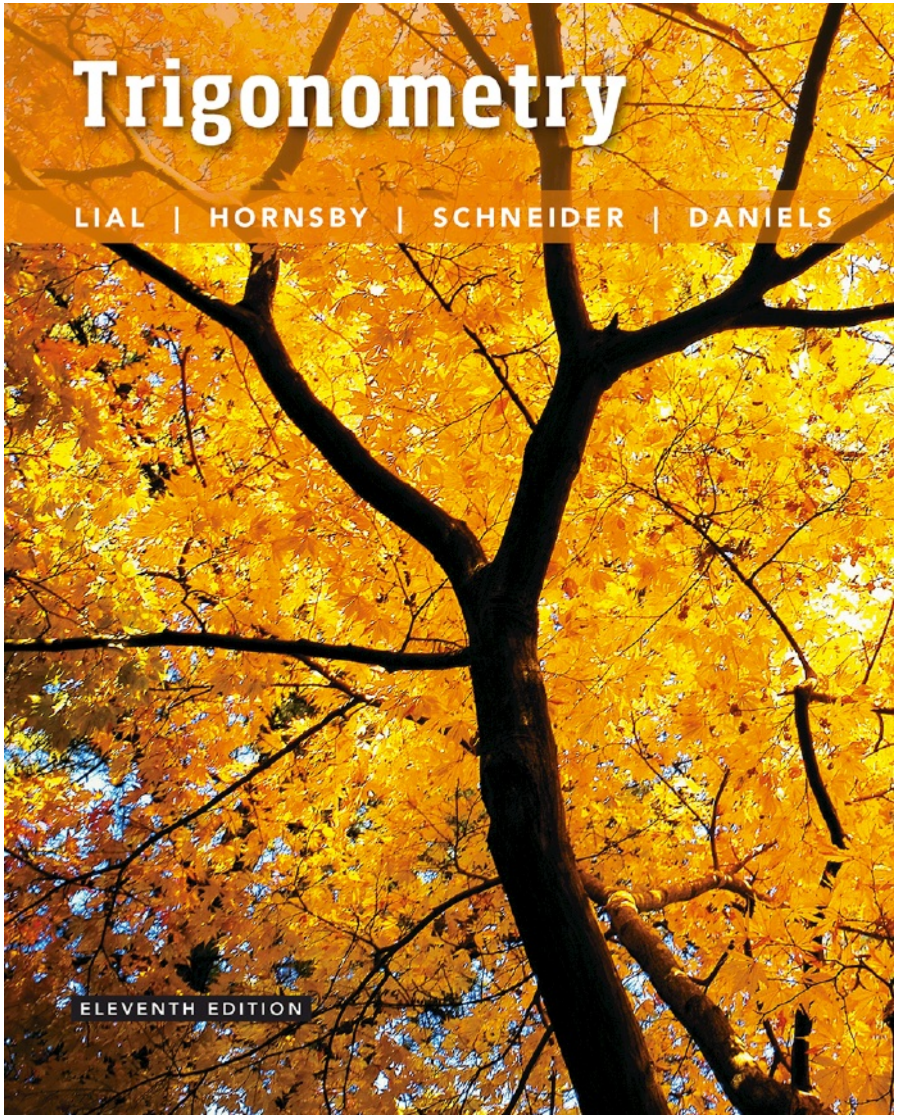
5

Trigonometric Identities



5.6 Half-Angle Identities

Half-Angle Identities ▪ Applications of the Half-Angle Identities ▪ Verifying an Identity

Half-Angle Identities

We can use the cosine sum identities to derive half-angle identities.

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}$$

$$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

Let $A = 2x$.

Choose the appropriate sign depending on the quadrant of $\frac{A}{2}$.

Half-Angle Identities

$$\cos 2x = 2\cos^2 x - 1$$

$$2\cos^2 x = \cos 2x + 1$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \text{Let } A = 2x.$$

Choose the appropriate sign depending on the quadrant of $\frac{A}{2}$.

Half-Angle Identities

There are three alternative forms for $\tan \frac{A}{2}$.

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\pm\sqrt{\frac{1-\cos A}{2}}}{\pm\sqrt{\frac{1+\cos A}{2}}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2\sin \frac{A}{2}\cos \frac{A}{2}}{2\cos^2 \frac{A}{2}} = \frac{\sin 2\left(\frac{A}{2}\right)}{1+\cos 2\left(\frac{A}{2}\right)} = \frac{\sin A}{1+\cos A}$$

Half-Angle Identities

From the identity $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$, we can also derive an equivalent identity.

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$$

Half-Angle Identities

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} \qquad \tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$$

Example 1 USING A HALF-ANGLE IDENTITY TO FIND AN EXACT VALUE

Find the exact value of cos 15° using the half-angle identity for cosine.

$$\cos 15^\circ = \cos \frac{1}{2}(30^\circ) = \sqrt{\frac{1+\cos 30^\circ}{2}}$$

Choose the positive square root.

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 2 USING A HALF-ANGLE IDENTITY TO FIND AN EXACT VALUE

Find the exact value of tan 22.5° using the identity

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}.$$

$$\tan 22.5^\circ=\tan\left(\frac{45^\circ}{2}\right)=\frac{\sin 45^\circ}{1+\cos 45^\circ}$$

$$=\frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}=\frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}$$

$$=\frac{\sqrt{2}}{2+\sqrt{2}}=\frac{\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$=\frac{2\sqrt{2}-2}{4-2}=\frac{2\left(\sqrt{2}-1\right)}{2}=\sqrt{2}-1$$

FINDING FUNCTION VALUES OF *s*/2 GIVEN INFORMATION ABOUT *s*

Given $\cos s = \frac{2}{3}$, with $\frac{3\pi}{2} < s < 2\pi$, find $\cos \frac{s}{2}$, $\sin \frac{s}{2}$, and $\tan \frac{s}{2}$.

The angle associated with $\frac{s}{2}$ lies in quadrant II since $\frac{3\pi}{2} < s < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{s}{2} < \pi$.

$\sin \frac{s}{2}$ is positive while $\cos \frac{s}{2}$ and $\tan \frac{s}{2}$ are negative.

Example 3 FINDING FUNCTION VALUES OF *s*/2 GIVEN INFORMATION ABOUT *s* (cont.)

$$\sin\frac{s}{2}=\sqrt{\frac{1-\cos s}{2}}=\sqrt{\frac{1-\frac{2}{3}}{2}}=\sqrt{\frac{\frac{1}{3}}{2}}=\sqrt{\frac{1}{6}}=\frac{\sqrt{6}}{6}$$

$$\cos\frac{s}{2}=-\sqrt{\frac{1+\cos s}{2}}=-\sqrt{\frac{1+\frac{2}{3}}{2}}=-\sqrt{\frac{\frac{5}{3}}{2}}=-\sqrt{\frac{5}{6}}=-\frac{\sqrt{30}}{6}$$

$$\tan\frac{s}{2}=\frac{\sin\frac{s}{2}}{\cos\frac{s}{2}}=\frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}}=-\frac{\sqrt{6}}{\sqrt{30}}=-\frac{\sqrt{180}}{30}=-\frac{\sqrt{5}}{5}$$

Example 4 SIMPLIFYING EXPRESSIONS USING THE HALF-ANGLE IDENTITIES

Simplify each expression.

(a) $\pm\sqrt{\dfrac{1+\cos 12x}{2}}$

This matches part of the identity for $\cos\dfrac{A}{2}$.

Substitute 12*x* for *A*: $\pm\sqrt{\dfrac{1+\cos 12x}{2}}=\cos\dfrac{12x}{2}=\cos 6x$

(b) $\dfrac{1-\cos 5\alpha}{\sin 5\alpha}=\tan\dfrac{5\alpha}{2}$

Example 5 VERIFYING AN IDENTITY

Verify that $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = 1+\sin x$ is an identity.

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 = \sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}$$

$$= \left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 1+\sin\left(2\cdot\frac{x}{2}\right)$$

$$= 1+\sin x$$