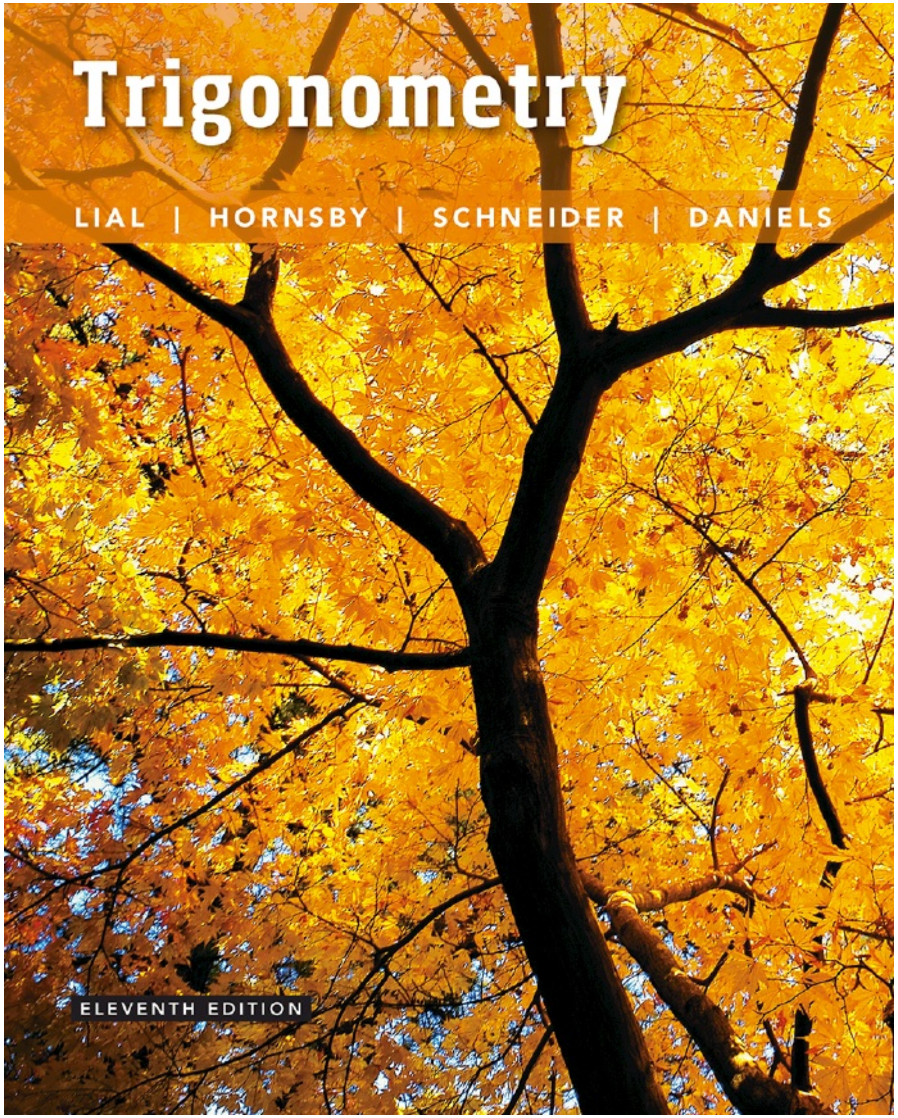
6

Inverse Circular Functions and Trigonometric Equations



6.1 Inverse Circular Functions

Inverse Functions ▪ Inverse Sine Function ▪ Inverse Cosine Function ▪ Inverse Tangent Function ▪ Other Inverse Circular Functions ▪ Inverse Function Values

Functions

Recall that for a function f, every element x in the domain corresponds to one and only one element y, or $f(x)$, in the range.

If a function is defined so that ***each range element is used only once,*** then it is called a **one-to-one function.**

Inverse Function

The **inverse function** of the one-to-one function *f* is defined as follows.

$$f^{-1} = \{(y, x) \mid (x, y) \text{ belongs to } f\}.$$

Caution

The –1 in f^{-1} is not an exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Review of Inverse Functions

- In a one-to-one function, each x-value correspond to only one y-value, and each y-value corresponds to only one x-value.
- If a function f is one-to-one, then f has an inverse function f^{-1}.
- The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
- The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

Review of Inverse Functions

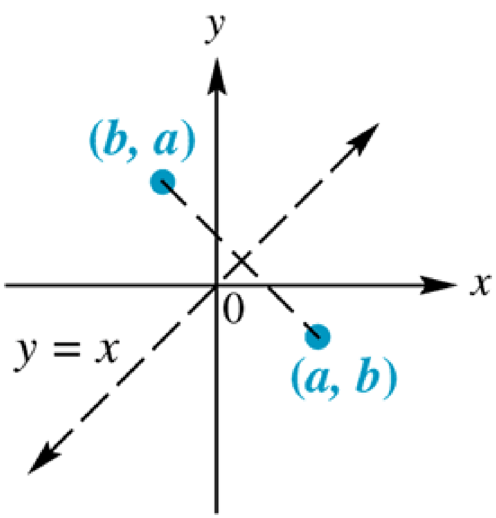
- To find $f^{-1}(x)$ from $f(x)$, follow these steps:

Step 1 Replace $f(x)$ with y and interchange x and y.

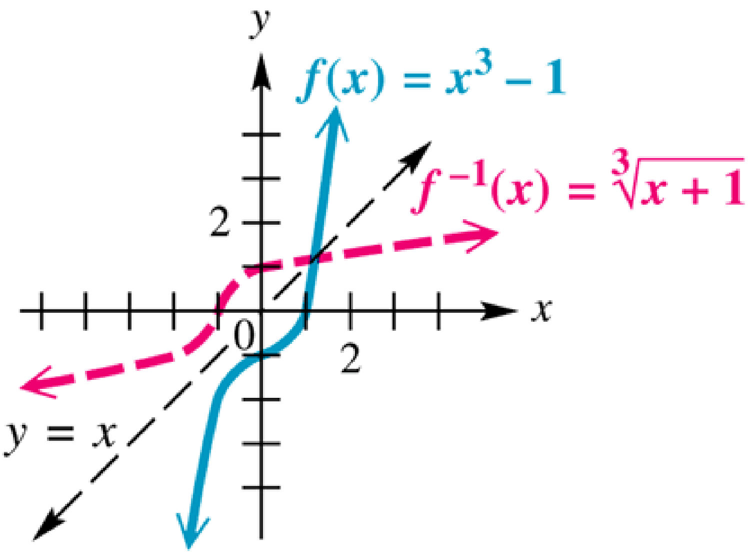
Step 2 Solve for y.

Step 3 Replace y with $f^{-1}(x)$.

Inverse Functions



(b, a) is the reflection of (a, b) across the line $y = x$.



The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

Inverse Sine Function

$y = \sin^{-1} x$ or $y = \arcsin x$ means that $x = \sin y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Think of* $y = \sin^{-1} x$ *or* $y = \arcsin x$ *as

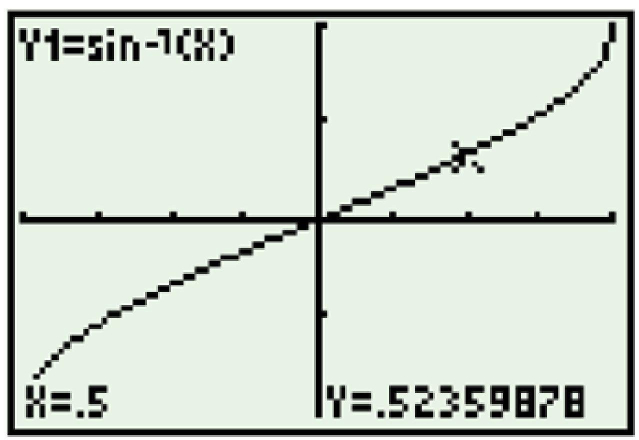
"y is the number (angle) in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x."

Find *y* in each equation.

(a) $y = \arcsin\frac{1}{2} \Rightarrow \sin y = \frac{1}{2}, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Since $\sin\frac{\pi}{6} = \frac{1}{2}$, $y = \frac{\pi}{6}$.

$\frac{\pi}{2}$

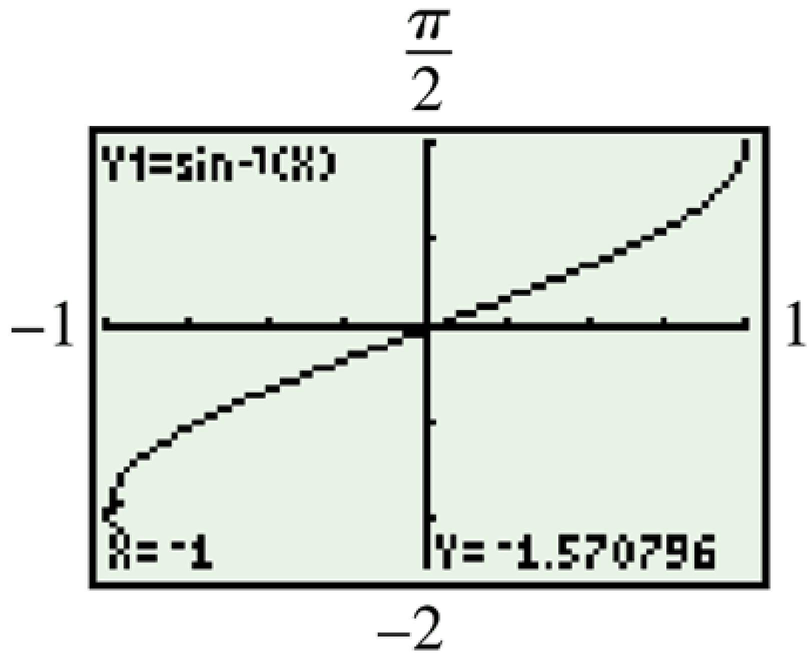
-1  1

$-\frac{\pi}{2}$

Example 1 FINDING INVERSE SINE VALUES (cont.)

(b) $y = \sin^{-1}(-1) \Rightarrow \sin y = -1, \; -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\text{Since } \sin\left(-\frac{\pi}{2}\right) = -1, \; y = -\frac{\pi}{2}.$$



Example 1 FINDING INVERSE SINE VALUES (cont.)

(c) $y = \sin^{-1}(-2)$

–2 is not in the domain of the inverse sine function, [–1, 1], so $\sin^{-1}(-2)$ does not exist.

A graphing calculator will give an error message for this input.

Caution

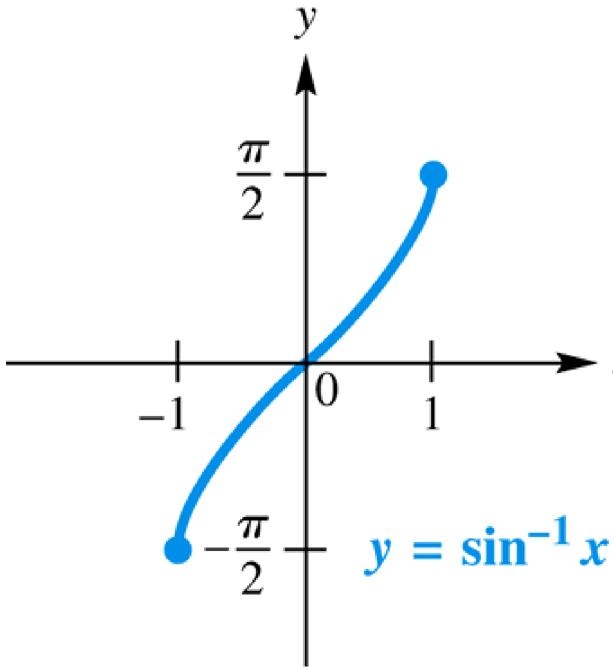
Be certain that the number given for an inverse function value is in the range of the particular inverse function being considered.

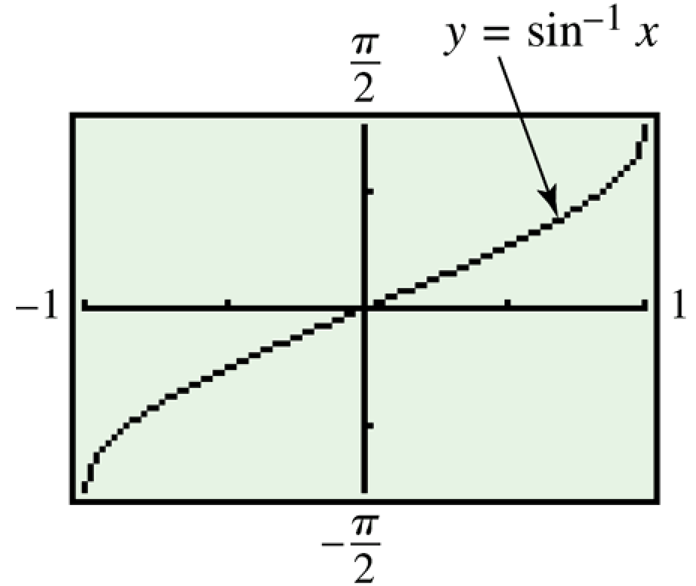
Inverse Sine Function
$y = \sin^{-1} x$ or $y = \arcsin x$

Domain: [–1, 1]

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

x	y
-1	$\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$





Inverse Sine Function $y = \sin^{-1} x$ or $y = \arcsin x$

- The inverse sine function is increasing and continuous on its domain $[-1, 1]$.
- Its x-intercept is 0, and its y-intercept is 0.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\sin^{-1}(-x) = -\sin^{-1}x$.

Inverse Cosine Function

$y = \cos^{-1} x$ or $y = \arccos x$ means that $x = \cos y$, for $0 \le y \le \pi$.

Think of $\boldsymbol{y = \cos^{-1} x}$ ***or*** $\boldsymbol{y = \arccos x}$ ***as***

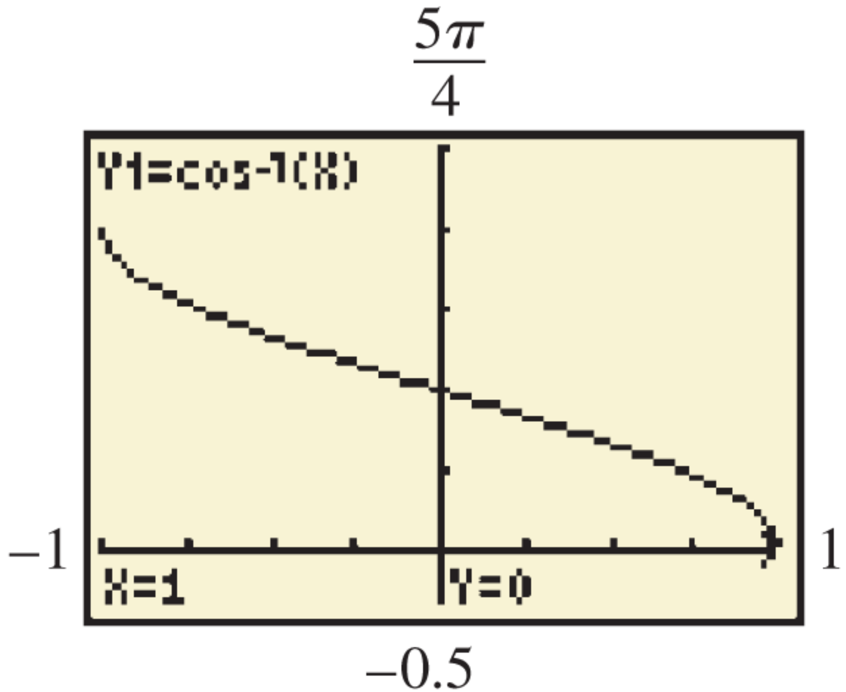
"y is the number (angle) in the interval $[0, \pi]$ ***whose cosine is x."***

Example 2 FINDING INVERSE COSINE VALUES

Find *y* in each equation.

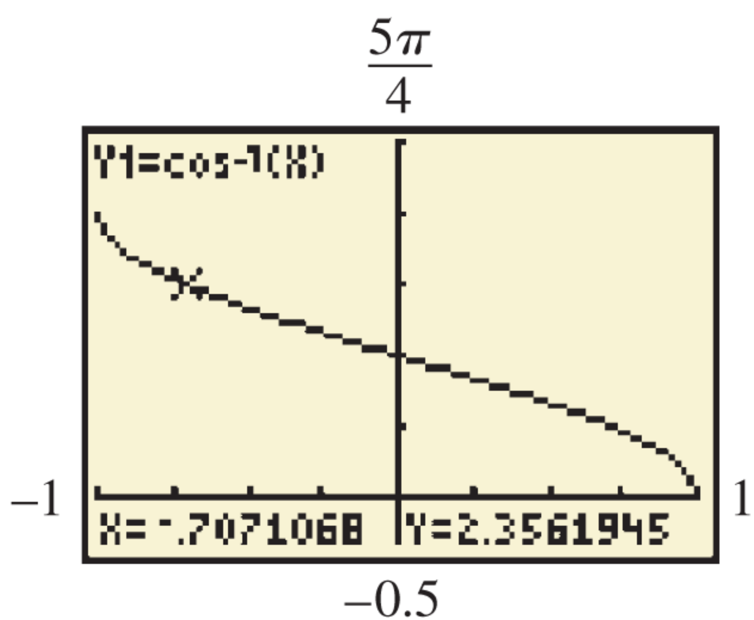
(a) $y = \arccos 1 \Rightarrow \cos y = 1,\ 0 \le y \le \pi$

$$\text{Since } \cos 0 = 1,\ y = 0.$$



(b) $y = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) \Rightarrow \cos y = -\frac{\sqrt{2}}{2},\ 0 \le y \le \pi$

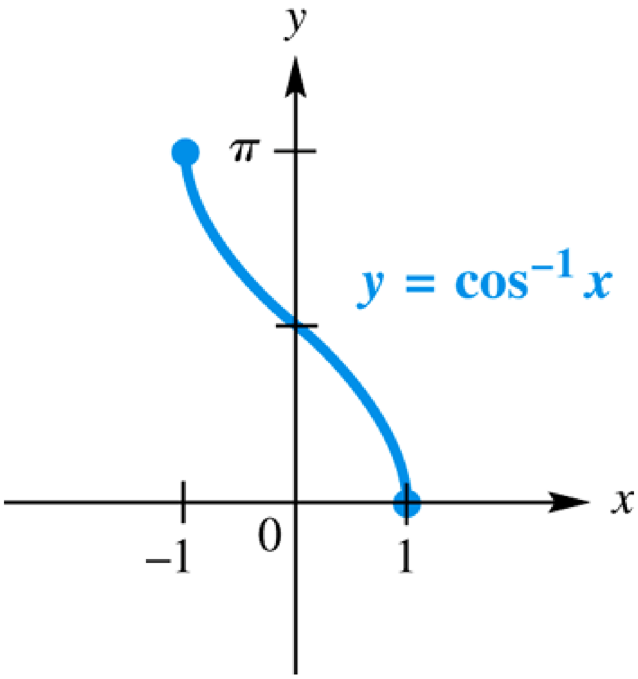
$$\text{Since } \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2},\ y = \frac{3\pi}{4}.$$

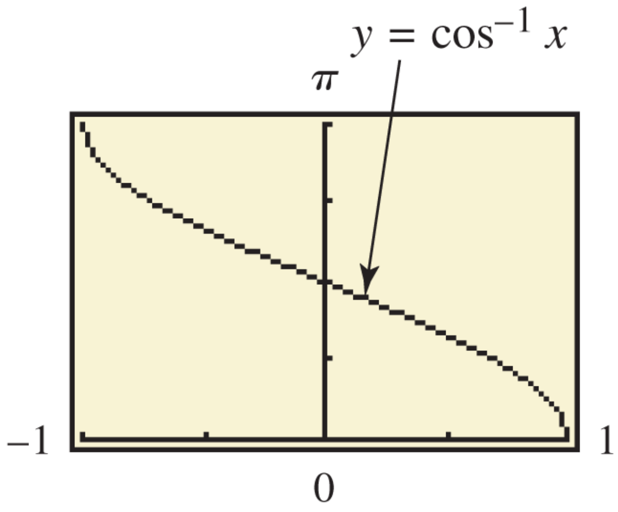


Inverse Cosine Function
$y = \cos^{-1} x$ or $y = \arccos x$

Domain: [–1, 1] Range: $[0, \pi]$

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0





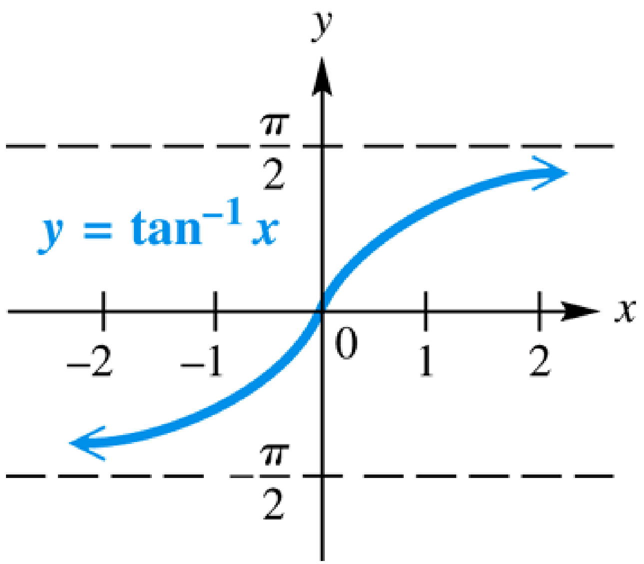
Inverse Cosine Function $y = \cos^{-1} x$ or $y = \arccos x$

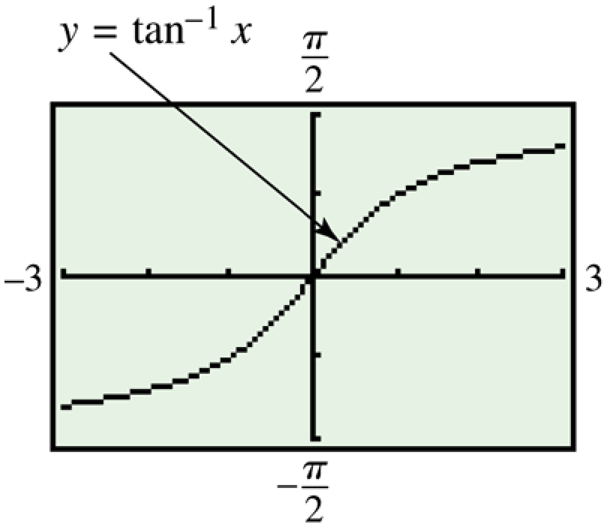
- The inverse cosine function is decreasing and continuous on its domain [–1, 1].
- Its *x*-intercept is 1, and its *y*-intercept is $\frac{\pi}{2}$.
- The graph is not symmetric with respect to the *y*-axis nor the origin.

Inverse Tangent Function $y = \tan^{-1} x$ or $y = \arctan x$

Domain: $(-\infty, \infty)$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

x	y
-1	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
0	0
$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$





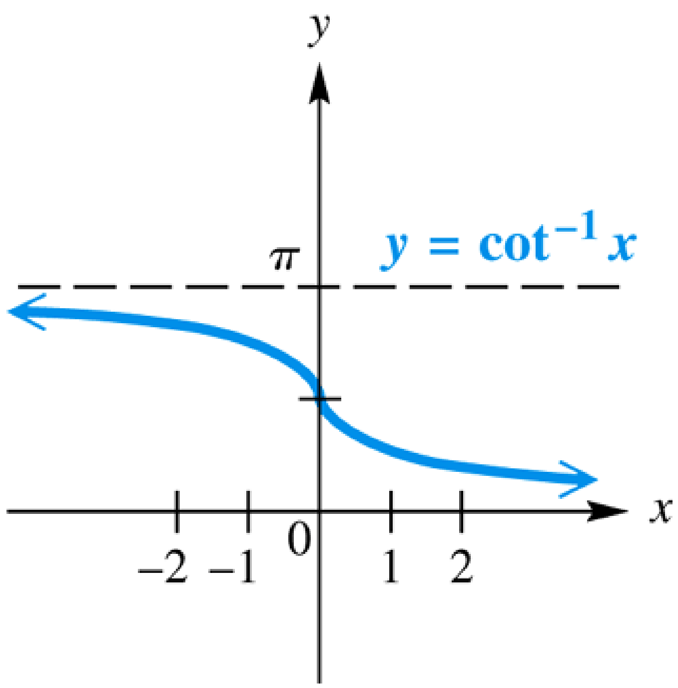
Inverse Tangent Function $y = \tan^{-1} x$ or $y = \arctan x$

- The inverse tangent function is increasing and continuous on its domain $(-\infty, \infty)$.
- Its x-intercept is 0, and its y-intercept is 0.
- Its graph is symmetric with respect to the origin so the function is an odd function.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

$y = \tan^{-1} x$ or $y = \arctan x$ means that $x = \tan y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Inverse Cotangent Function $y = \cot^{-1} x$ or $y = \operatorname{arccot} x$

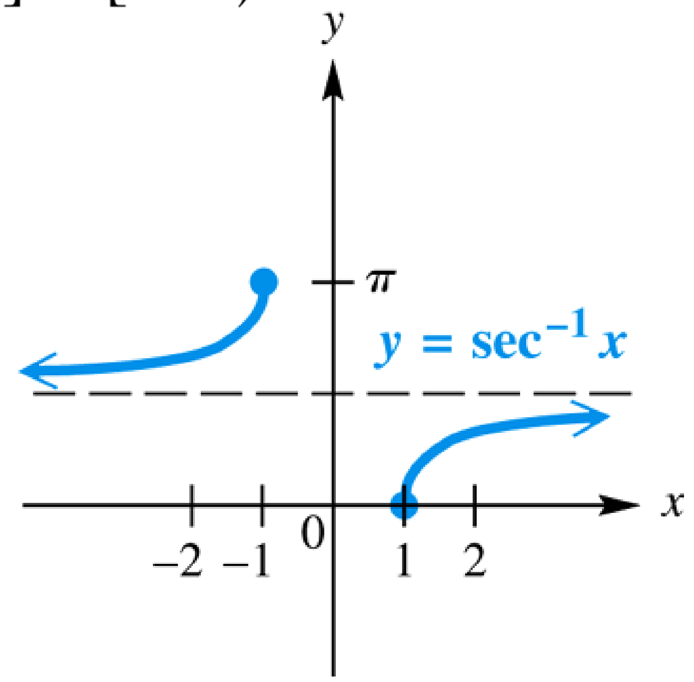
Domain: $(-\infty, \infty)$ Range: $(0, \pi)$



$y = \cot^{-1} x$ or $y = \operatorname{arccot} x$ means that $x = \cot y$ for $0 < y < \pi$.

Inverse Secant Function
$y = \sec^{-1} x$ or $y = \text{arcsec } x$

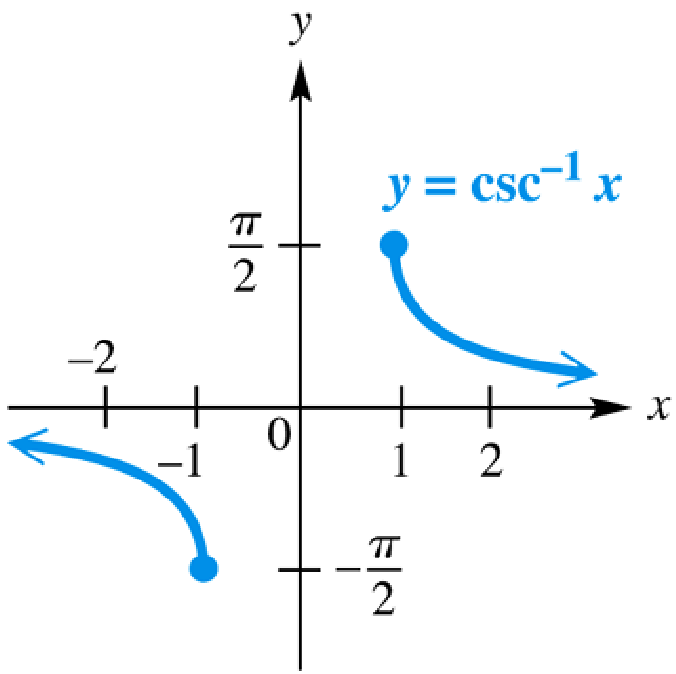
Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $[0, \pi]$, $y \neq \frac{\pi}{2}$



$y = \sec^{-1} x$ or $y = \text{arc}\sec x$ means that $x = \sec y$ for $0 \le y \le \pi$, $y \neq \frac{\pi}{2}$.

Inverse Cosecant Function
$y = \csc^{-1} x$ or $y = \text{arccsc } x$

Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $y \neq 0$



$y = \csc^{-1} x$ or $y = \text{arc}\csc x$ means that $x = \csc y$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$.

Inverse Function Values

Inverse Function	Domain	Range	
		Interval	Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	I and IV

Example 3 FINDING INVERSE FUNCTION VALUES (DEGREE-MEASURED ANGLES)

Find the *degree measure* of θ in the following.

(a) $\theta = \arctan 1$

θ must be in (–90°, 90°), but since 1 is positive, θ must be in quadrant I.

$$\theta = \arctan 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$$

(b) $\theta = \sec^{-1} 2$

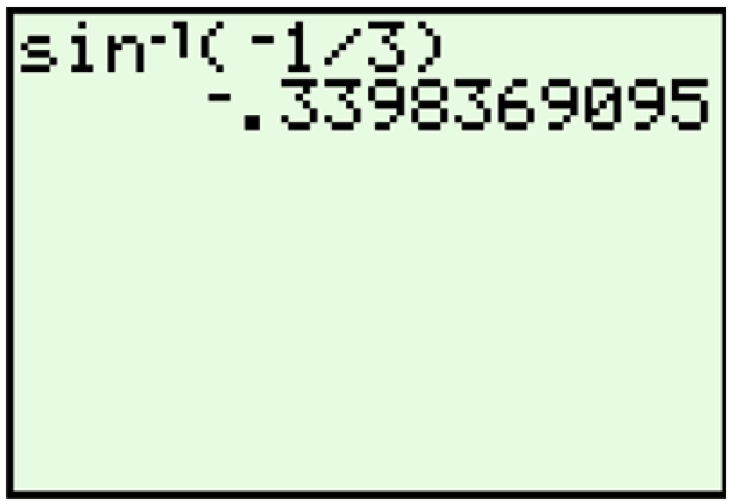
θ must be in quadrant I or II, but since 2 is positive, θ must be in quadrant I.

$$\theta = \sec^{-1} 2 \Rightarrow \sec\theta = 2 \Rightarrow \theta = 60^\circ$$

FINDING INVERSE FUNCTION VALUES WITH A CALCULATOR

(a) Find y in radians if $y = \csc^{-1}(-3)$.

With the calculator in radian mode, enter $y = \csc^{-1}(-3)$ as $\sin^{-1}\left(-\frac{1}{3}\right)$.



$$y \approx -0.3398969095$$

Example 4 FINDING INVERSE FUNCTION VALUES WITH A CALCULATOR (continued)

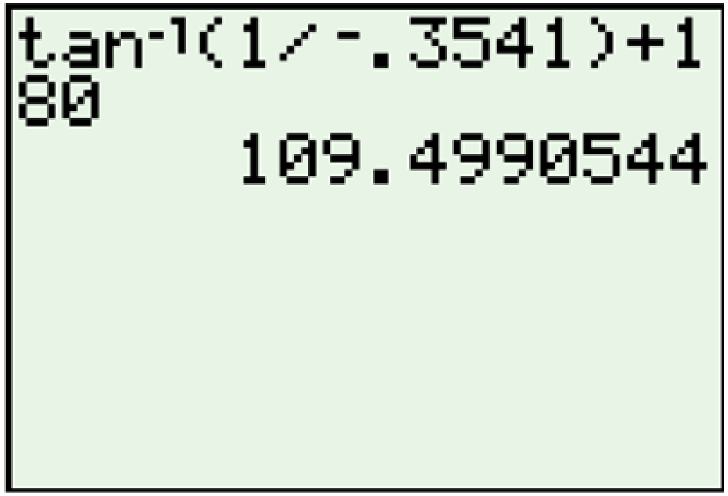
(b) Find θ in degrees if θ = arccot(–0.3541).

Set the calculator to degree mode. A calculator gives the inverse cotangent value of a negative number as a quadrant IV angle.

The restriction on the range of arccotangent implies that the angle must be in quadrant II, so enter arccot(–0.3541) as

$$\tan^{-1}\left(\frac{1}{-.3541}\right)+180°.$$

Example 4 FINDING INVERSE FUNCTION VALUES WITH A CALCULATOR (continued)



$$\theta \approx 109.4990544°$$

Caution

Be careful when using a calculator to evaluate the inverse cotangent of a negative quantity.

Enter the inverse tangent of the *reciprocal* of the negative quantity, which returns an angle in quadrant IV.

Since inverse cotangent is negative in quadrant II, adjust your calculator result by adding 180° or π accordingly.

▶ Example 5 FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

Evaluate each expression without a calculator.

(a) $\sin\left(\tan^{-1}\frac{3}{2}\right)$

$$\text{Let } \theta = \tan^{-1}\frac{3}{2}, \text{ so } \tan\theta = \frac{3}{2}.$$

Since arctan is defined only in quadrants I and IV, and $\frac{3}{2}$ is positive, θ is in quadrant I.

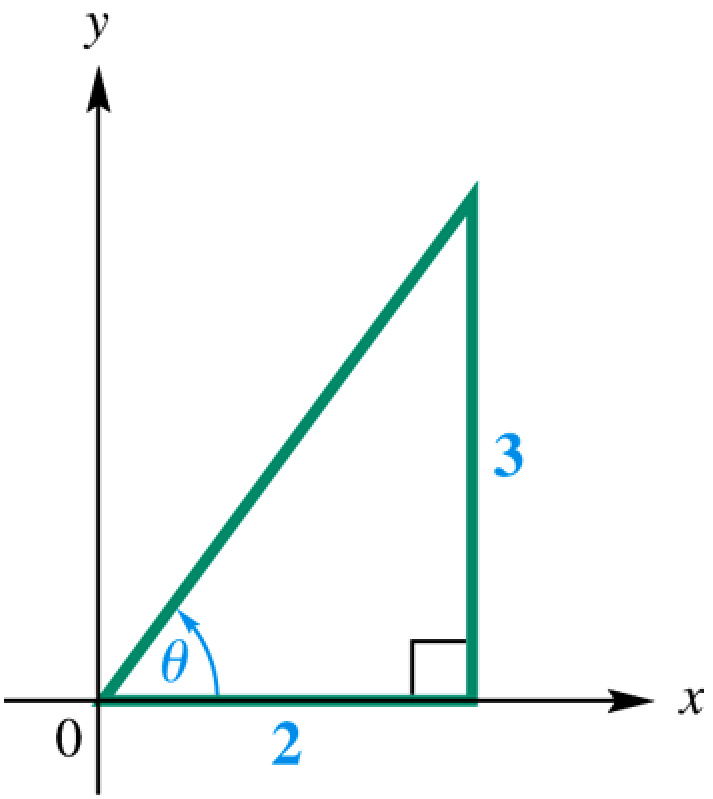
Sketch θ in quadrant I, and label the triangle.

Example 5 FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS (continued)

$$\sqrt{2^2+3^2}=\sqrt{13}$$

$$\sin\left(\tan^{-1}\frac{3}{2}\right)=\sin\theta$$

$$=\frac{3}{\sqrt{13}}=\frac{3\sqrt{13}}{13}$$



$\theta = \tan^{-1}\frac{3}{2}$

Example 5 FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS (continued)

(b) $\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$

$$\text{Let } A = \cos^{-1}\left(-\frac{5}{13}\right), \text{ so } \cos A = -\frac{5}{13}.$$

Since arccos is defined only in quadrants I and II, and $-\frac{5}{13}$ is negative, θ is in quadrant II.

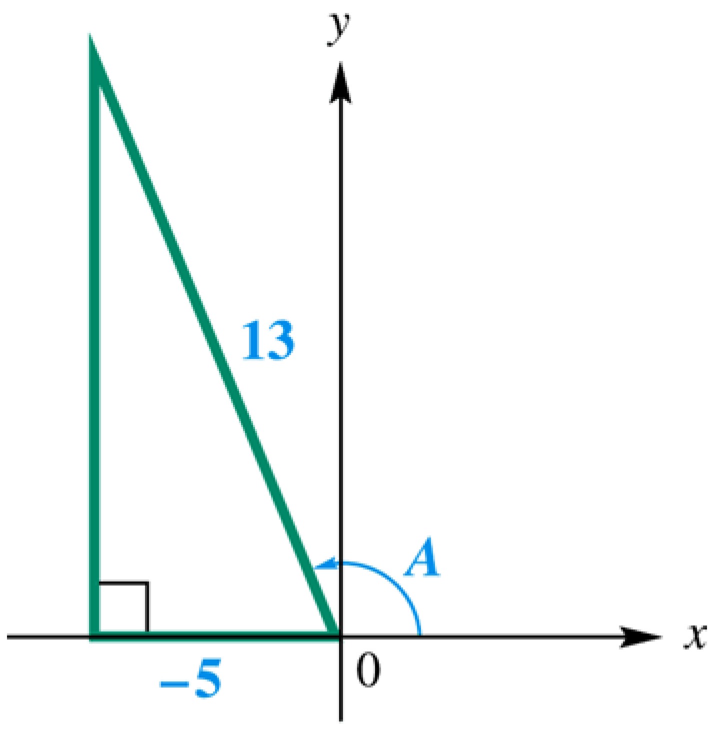
Sketch A in quadrant II, and label the triangle.

Example 5 FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS (continued)

$$\sqrt{13^2 - (-5)^2} = 12$$

$$\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) = \tan A$$

$$= \frac{12}{-5} = -\frac{12}{5}$$



$A = \cos^{-1}\left(-\frac{5}{13}\right)$

Example 6(a) FINDING FUNCTION VALUES USING IDENTITIES

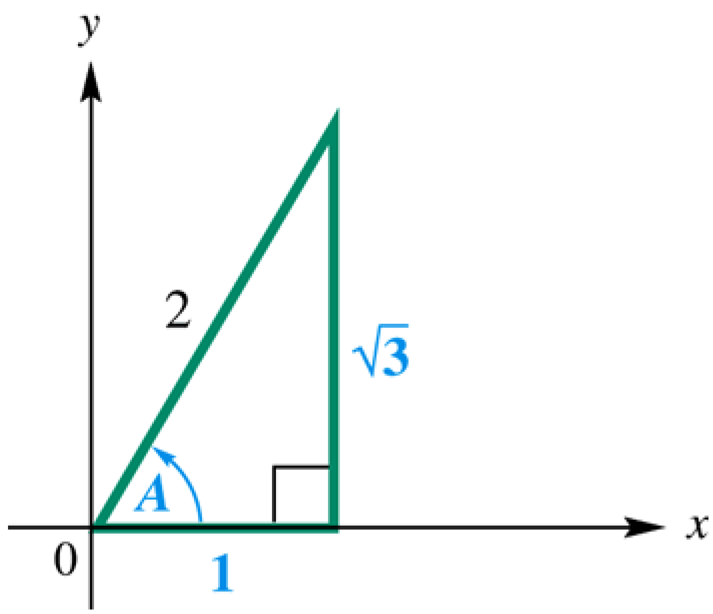
Evaluate the expression without a calculator.

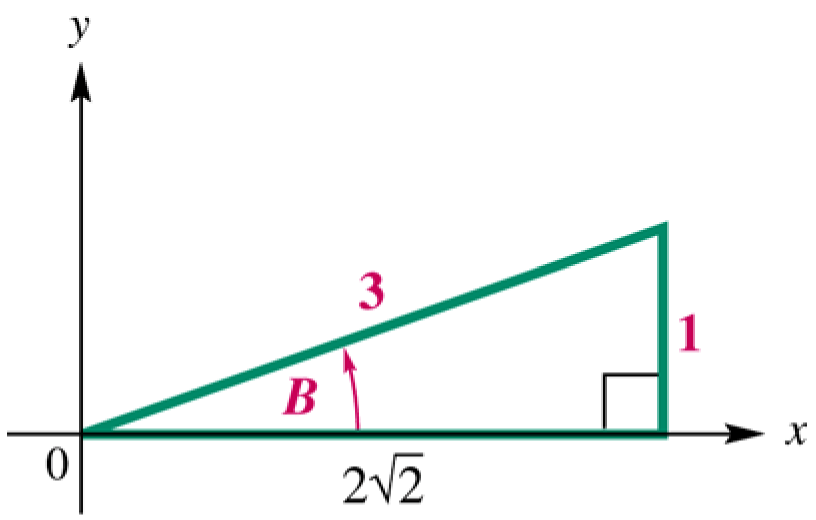
$$\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right)$$

Let $A = \arctan\sqrt{3}$ and $B = \arcsin\frac{1}{3}$, so $\tan A = \sqrt{3}$ and $\sin B = \frac{1}{3}$.

Example 6(a) FINDING FUNCTION VALUES USING IDENTITIES (continued)

Sketch both *A* and *B* in quadrant I. Use the Pythagorean theorem to find the missing side.





$$\sin A = \frac{\sqrt{3}}{2}, \cos A = \frac{1}{2} \qquad \sin B = \frac{1}{3}, \cos B = \frac{2\sqrt{2}}{3}$$

Example 6(a) FINDING FUNCTION VALUES USING IDENTITIES (continued)

Use the cosine sum identity:

$$\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)=\cos(A+B)$$

$$=\cos A\cos B-\sin A\,\mathrm{Sin}\,B$$

$$=\frac{1}{2}\cdot\frac{2\sqrt{2}}{3}-\frac{\sqrt{3}}{2}\cdot\frac{1}{3}$$

$$=\frac{2\sqrt{2}-\sqrt{3}}{6}$$

Example 6(b) FINDING FUNCTION VALUES USING IDENTITIES

Evaluate the expression without a calculator.

$$\tan\left(2\arcsin\frac{2}{5}\right)$$

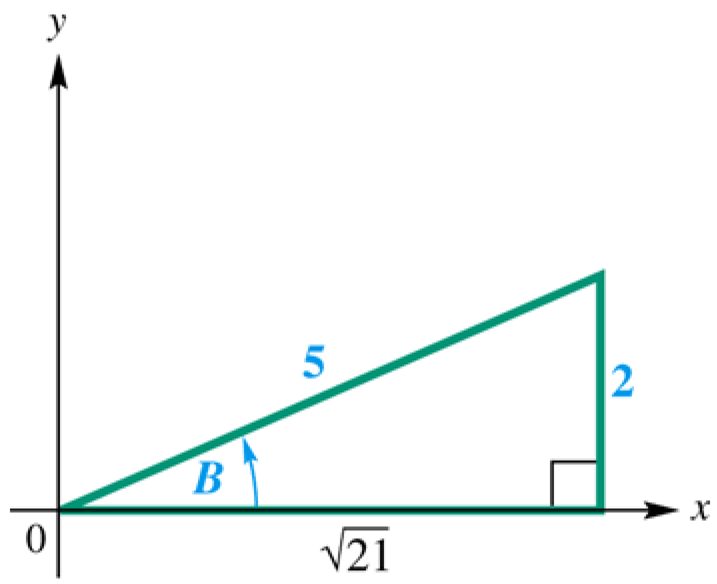
Let $B = \arcsin\frac{2}{5}$, so $\sin B = \frac{2}{5}$.

Use the double-angle identity:

$$\tan 2B = \frac{2\tan B}{1-\tan^2 B}$$

Example 6(b) FINDING FUNCTION VALUES USING IDENTITIES (continued)

Sketch *B* in quadrant I. Use the Pythagorean theorem to find the missing side.



$$\tan B = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Example 6(b) FINDING FUNCTION VALUES USING IDENTITIES (continued)

$$\tan 2B = \frac{2\tan B}{1-\tan^2 B}$$

$$= \frac{2\left(\frac{2\sqrt{21}}{21}\right)}{1-\left(\frac{2\sqrt{21}}{21}\right)^2} = \frac{\frac{4\sqrt{21}}{21}}{1-\frac{4\cdot 21}{21^2}}$$

$$= \frac{\frac{4\sqrt{21}}{21}}{1-\frac{4}{21}} = \frac{\frac{4\sqrt{21}}{21}}{\frac{17}{21}}$$

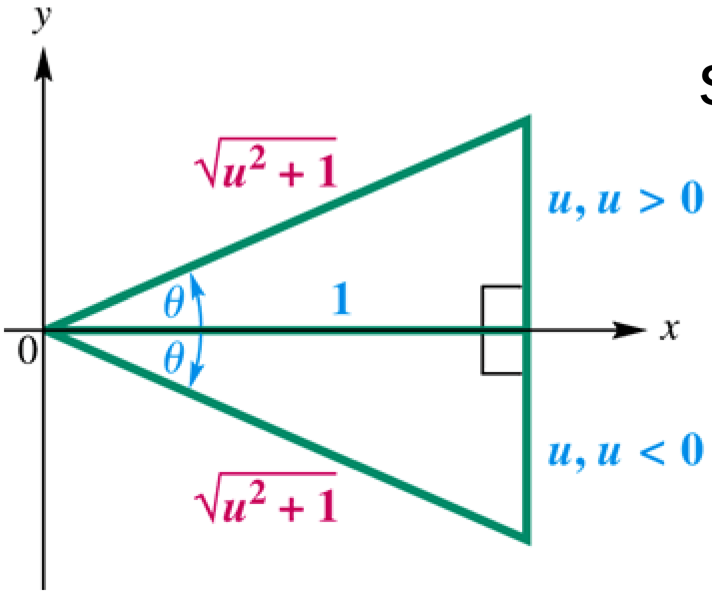
$$= \frac{4\sqrt{21}}{17}$$

WRITING FUNCTION VALUES IN TERMS OF *u*

Write $\sin\left(\tan^{-1} u\right)$ as an algebraic expression in *u*.

Let $\theta = \tan^{-1} u$, so $\tan\theta = u$.

u may be positive or negative. Since $-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}$, sketch *θ* in quadrants I and IV.



$$\sin\left(\tan^{-1} u\right) = \sin\theta = \frac{u}{\sqrt{u^2+1}}$$

$$= \frac{u\sqrt{u^2+1}}{u^2+1}$$

Example 7(b) WRITING FUNCTION VALUES IN TERMS OF *u*

Write $\cos\left(2\sin^{-1}u\right)$ as an algebraic expression in *u*.

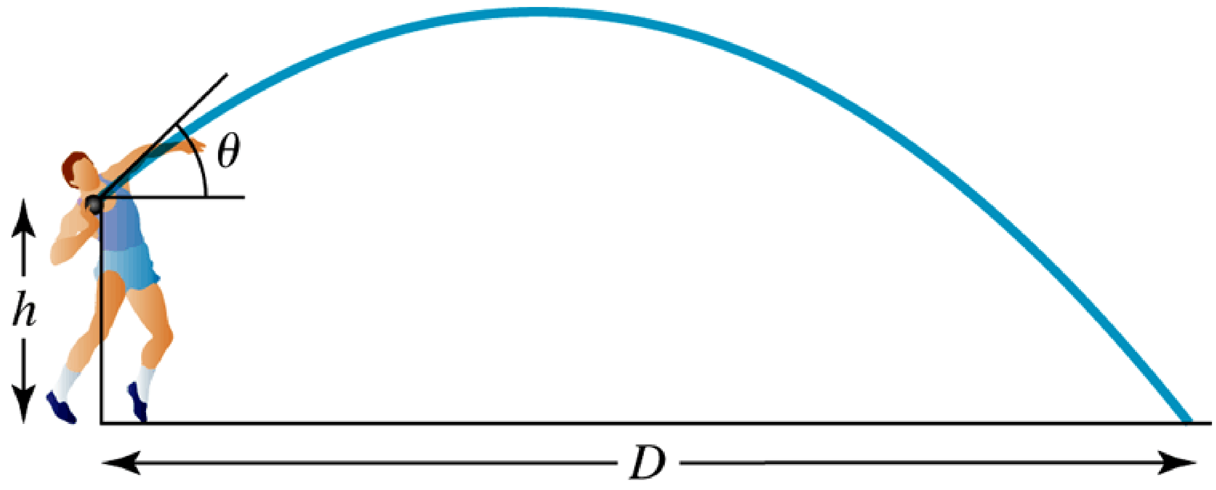
Let $\theta = \sin^{-1}u$, so $\sin\theta = u$.

Use the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\begin{aligned}\cos\left(2\sin^{-1}u\right) &= \cos 2\theta = 1 - 2\sin^2\theta \\ &= 1 - 2u^2\end{aligned}$$

Example 8 FINDING THE OPTIMAL ANGLE OF ELEVATION OF A SHOT PUT

The optimal angle of elevation θ that a shot-putter should aim for in order to throw the greatest distance depends on the velocity v of the throw and the initial height h of the shot.*



**Source*: Townend, M. S., *Mathematics in Sport*, Chichester, Ellis Horwood Limited.)

Example 8 FINDING THE OPTIMAL ANGLE OF ELEVATION OF A SHOT PUT (continued)

One model for θ that achieves this greatest distance is

$$\theta = \arcsin\left(\sqrt{\frac{v^2}{2v^2 + 64h}}\right)$$

Suppose a shot-putter can consistently throw the steel ball with $h = 6.6$ ft and $v = 42$ ft per sec. At what angle should he release the ball to maximize distance?

$$\theta = \arcsin\left(\sqrt{\frac{42^2}{2\left(42^2\right) + 64(6.6)}}\right) \approx 42^\circ$$