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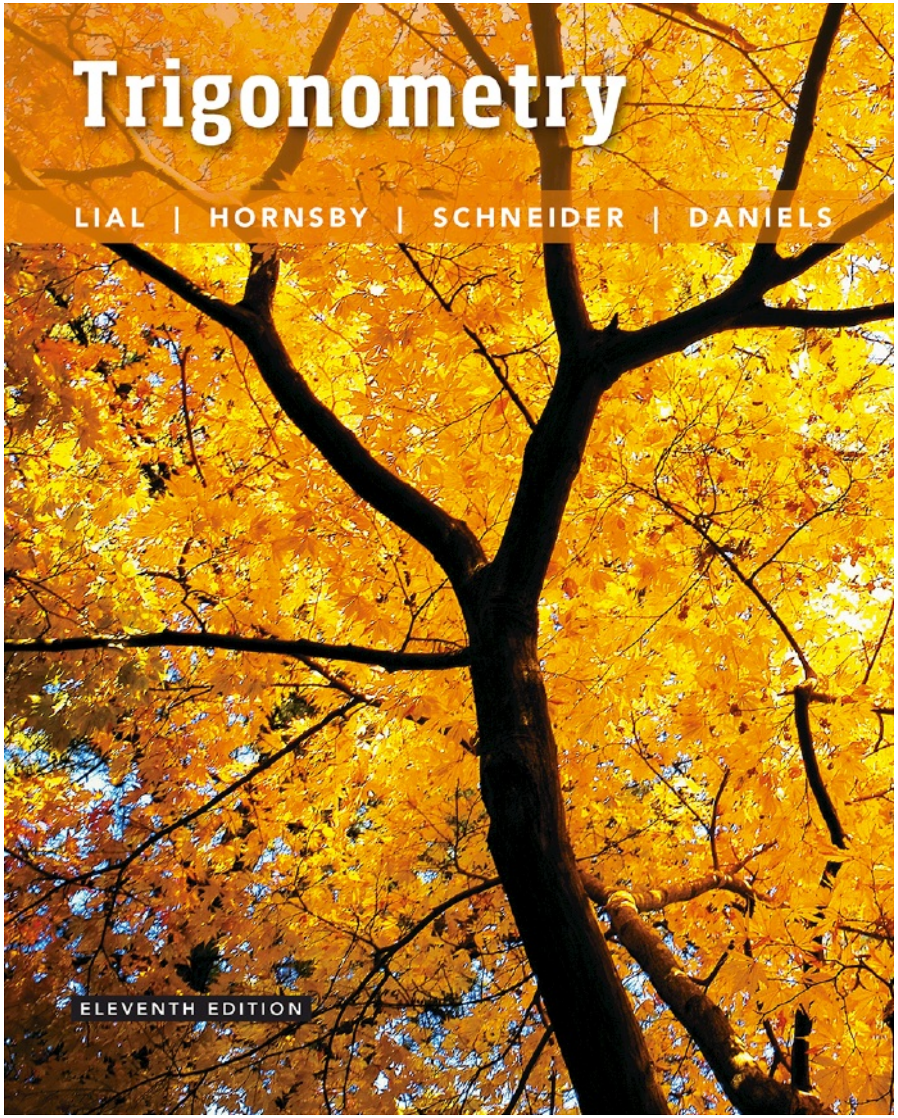
Inverse Circular Functions and Trigonometric Equations



6.2 Trigonometric Equations

Linear Methods ▪ Zero-Factor Property ▪ Quadratic Methods ▪ Trigonometric Identity Substitutions ▪ Application

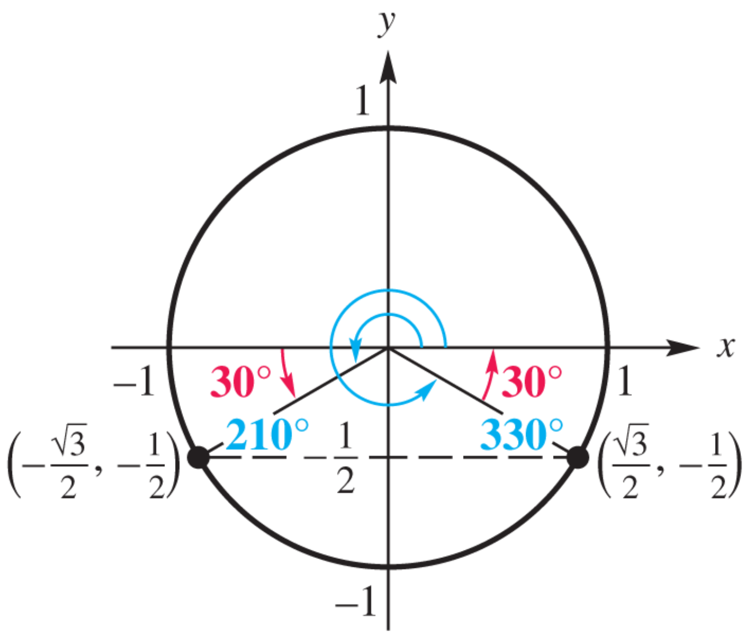
Example 1(a) SOLVING A TRIGONOMETRIC EQUATION BY LINEAR METHODS

Solve $2\sin\theta + 1 = 0$ over the interval $[0, 360°)$.

$$2\sin\theta + 1 = 0 \Rightarrow 2\sin\theta = -1 \Rightarrow \sin\theta = -\frac{1}{2}$$

Sin *θ* is negative in quadrants III and IV.

The reference angle is 30° because $\sin 30° = \frac{1}{2}$.



Solution set: {210°,330°}

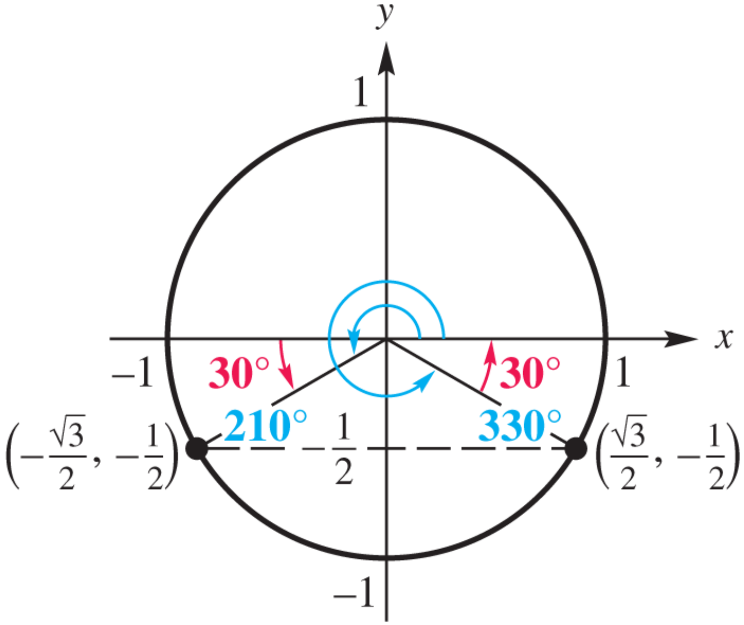
To find all solutions, add integer multiples of the period of the sine function to each solution found above.

Example 1(b) SOLVING A TRIGONOMETRIC EQUATION BY LINEAR METHODS

Solve the equation $2\sin\theta + 1 = 0$ for all solutions.

To find all solutions, add integer multiples of the period of the sine function to each solution found in part (a). The solutions set is written as follows.

$$\{210^\circ + 360^\circ n,\ 330^\circ + 360^\circ n, \text{ where } n \text{ is any integer}\}$$



Example 2 SOLVING A TRIGONOMETRIC EQUATION (ZERO-FACTOR PROPERTY)

Solve $\sin\theta\tan\theta = \sin\theta$ over the interval $[0, 360°)$.

$$\sin\theta\tan\theta = \sin\theta$$

$$\sin\theta\tan\theta - \sin\theta = 0$$ Subtract sin *θ*.

$$\sin\theta(\tan\theta - 1) = 0$$ Factor out sin *θ*.

$$\sin\theta = 0 \quad \text{or} \quad \tan\theta - 1 = \theta \Rightarrow \tan\theta = 1$$

Zero-factor property

$$\theta = 0° \text{ or } \theta = 180° \text{ or } \quad \theta = 45° \text{ or } \theta = 225°$$

Solution set: {0°, 45°, 180°, 225°}

Caution

Trying to solve the equation in **Example 2** by dividing each side by sin θ would lead to tan $\theta = 1$, which would give $\theta = 45°$ or $\theta = 225°$.

The missing solutions are the ones that make the divisor, sin θ, equal 0.

For this reason, avoid dividing by a variable expression.

Example 3 SOLVING A TRIGONOMETRIC EQUATION (ZERO-FACTOR PROPERTY)

Solve $\tan^2 x + \tan x - 2 = 0$ over the interval $[0, 2\pi)$.

$$\tan^2 x + \tan x - 2 = 0$$

$$(\tan x - 1)(\tan x + 2) = 0 \qquad \text{Factor.}$$

$$\tan x - 1 = 0 \quad \text{or} \quad \tan x + 2 = 0 \qquad \text{Zero-factor property}$$

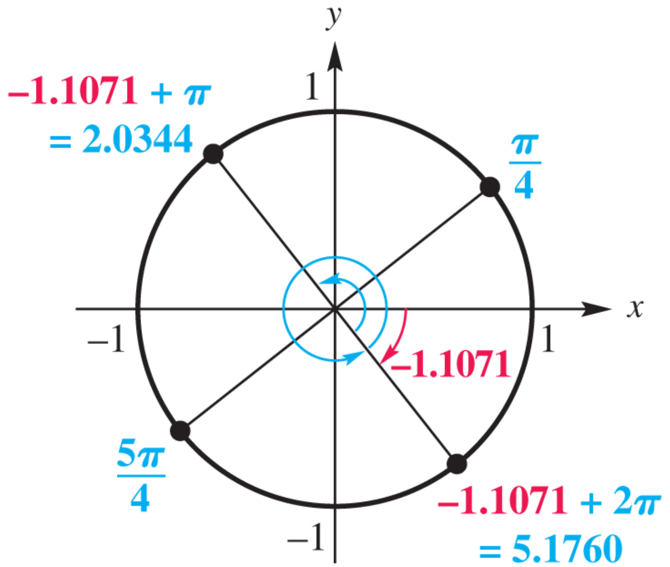
$$\tan x = 1 \quad \text{or} \quad \tan x = -2$$

The solutions for tan x = 1 over the interval $[0, 2\pi)$ are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Example 3 SOLVING A TRIGONOMETRIC EQUATION BY FACTORING (continued)

Based on the range of the inverse tangent function, $\tan^{-1}(-2) \approx -1.1071487$ a number in quadrant IV.

Since we want solutions over the interval $[0, 2\pi)$, add π to –1.1071487, and then add 2π.



The solutions shown in blue represent angle measures, in radians, ***and*** their intercepted arc lengths on the unit circle.

Solution set: $\left\{ \underbrace{\frac{\pi}{4}, \frac{5\pi}{4}}_{\text{Exact values}}, \underbrace{2.0344, 5.1760}_{\text{Approximate values to four decimal places}} \right\}$

Example 4 SOLVING A TRIGONOMETRIC EQUATION USING THE QUADRATIC FORMULA

Find all solutions of cot *x*(cot *x* + 3) = 1. Write the solution set.

$$\cot x(\cot x + 3) = 1$$

$$\cot^2 x + 3\cot x - 1 = 0$$ Write the equation in quadratic form.

Use the quadratic formula with *a* = 1, *b* = 3, and *c* = –1 and cot *x* as the variable.

$$\cot x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)} = \frac{-3 \pm \sqrt{13}}{2}$$

Example 4 SOLVING A TRIGONOMETRIC EQUATION USING THE QUADRATIC FORMULA (continued)

$$\cot x \approx -3.302775628 \quad \text{or} \quad \cot x \approx 0.3027756377$$

$$x \approx \cot^{-1}(-3.302775628) \quad \text{or} \quad x \approx \cot^{-1}(0.3027756377)$$

$$x \approx \tan^{-1}\left(\frac{1}{-3.302775628}\right) + \pi \quad \text{or} \quad x \approx \tan^{-1}\left(\frac{1}{0.3027756377}\right)$$

$$x \approx 2.847591352 \quad \text{or} \quad x \approx 1.276795025$$

To find *all* solutions, add integer multiples of the period of tangent, π. The solution set is

$$\{2.8476 + n\pi, 1.2768 + n\pi, \text{ where } n \text{ is any integer}\}$$

Example 5 SOLVING A TRIGONOMETRIC EQUATION BY SQUARING

Solve $\tan x + \sqrt{3} = \sec x$ over the interval $[0, 2\pi)$.

$$\left(\tan x + \sqrt{3}\right)^2 = \sec^2 x$$ Square each side.

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x$$ Pythagorean identity

$$2\sqrt{3}\tan x = -2$$ Subtract $3 + \tan^2 x$.

$$\tan x = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$ Divide by $2\sqrt{3}$, then rationalize the denominator.

Two possible solutions are $\dfrac{5\pi}{6}$ or $\dfrac{11\pi}{6}$.

Example 5 SOLVING A TRIGONOMETRIC EQUATION BY SQUARING (continued)

Since the solution was found by squaring both sides of an equation, we must check that each proposed solution is a solution of the original equation.

$$\tan\frac{5\pi}{6}+\sqrt{3}\stackrel{?}{=}\sec\frac{5\pi}{6}$$

$$-\frac{\sqrt{3}}{3}+\sqrt{3}\stackrel{?}{=}-\frac{2\sqrt{3}}{3}$$

$$\frac{2\sqrt{3}}{3}\neq-\frac{2\sqrt{3}}{3}$$

$\frac{5\pi}{6}$ is not a solution.

$$\tan\frac{11\pi}{6}+\sqrt{3}\stackrel{?}{=}\sec\frac{11\pi}{6}$$

$$-\frac{\sqrt{3}}{3}+\sqrt{3}\stackrel{?}{=}\frac{2\sqrt{3}}{3}$$

$$\frac{2\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$$

$\frac{11\pi}{6}$ is a solution.

Solution set: $\left\{\frac{11\pi}{6}\right\}$

Solving a Trigonometric Equation

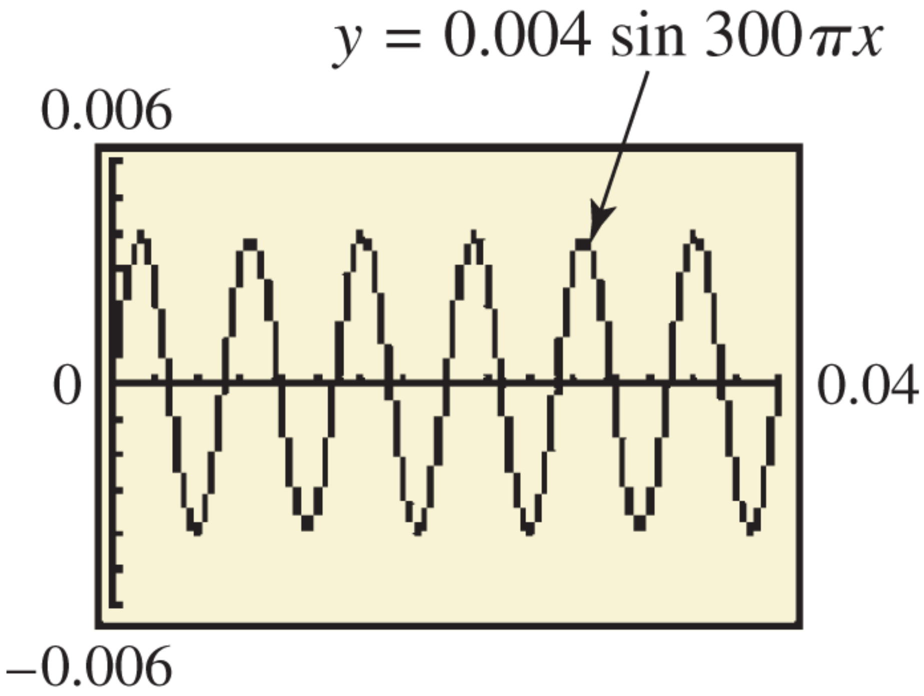
- Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
- If only one trigonometric function is present, first solve the equation for that function.
- If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.

Solving a Trigonometric Equation

- If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
- Try using identities to change the form of the equation. It may be helpful to square each side of the equation first. In this case, check for extraneous solutions.

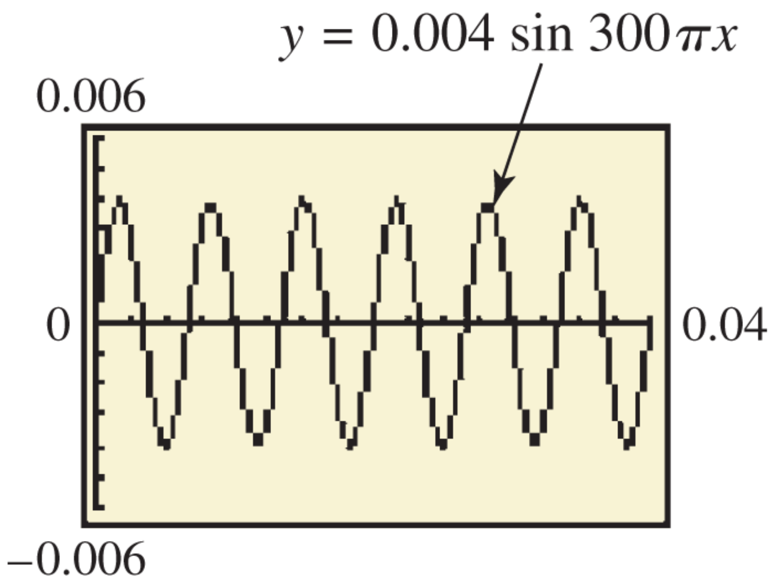
Example 6 DESCRIBING A MUSICAL TONE FROM A GRAPH

A basic component of music is a pure tone. The graph below models the sinusoidal pressure $y = P$ in pounds per square foot from a pure tone at time $x = t$ in seconds.



Example 6a DESCRIBING A MUSICAL TONE FROM A GRAPH (continued)

The frequency of a pure tone is often measured in hertz. One hertz is equal to one cycle per second and is abbreviated Hz. What is the frequency *f*, in hertz, of the pure tone shown in the graph?



From the graph, we see that there are 6 cycles in 0.04 sec.

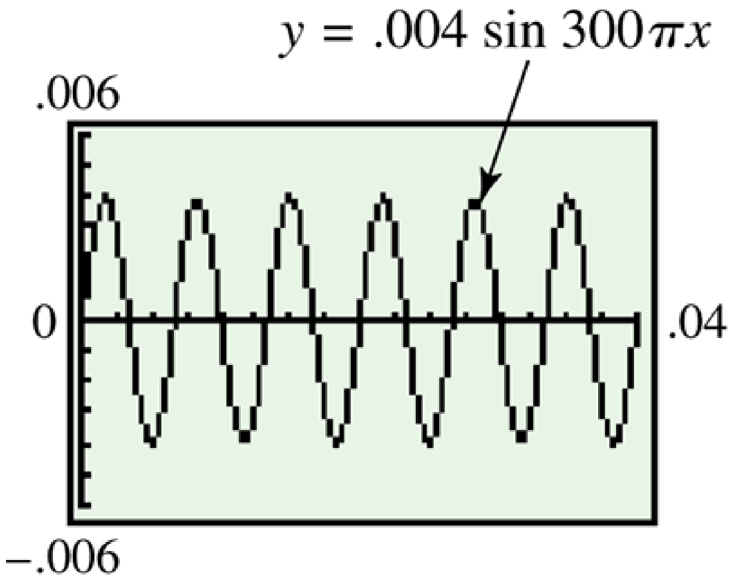
$$\frac{6}{0.04} = 150 \text{ cycles per sec}$$

The frequency *f* is 150 Hz.

Example 6b DESCRIBING A MUSICAL TONE FROM A GRAPH (continued)

The time for the tone to produce one complete cycle is called the **period.**

Approximate the period T, in seconds, of the pure tone.

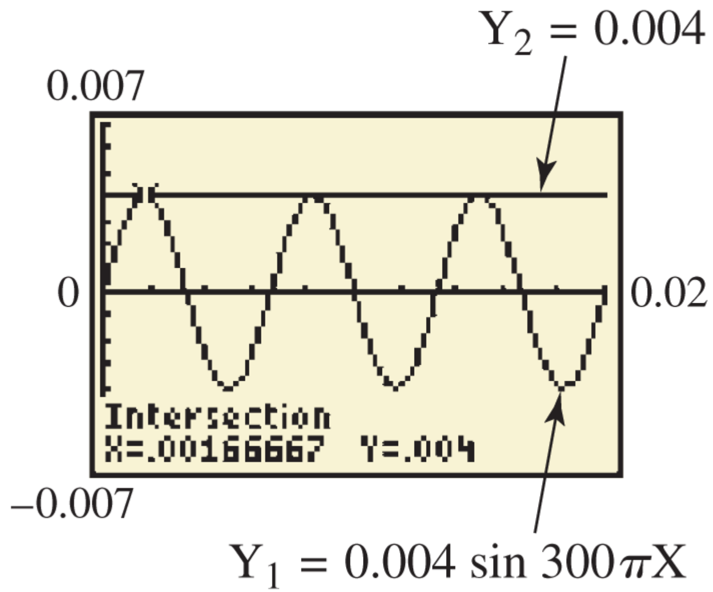


Six periods cover a time of 0.04 sec. One period would be equal to

$$T = \frac{0.04}{6} = \frac{1}{150}, \text{ or } 0.00\overline{6} \text{ sec.}$$

DESCRIBING A MUSICAL TONE FROM A GRAPH (continued)

An equation for the graph is $y = 0.004 \sin 300\pi x$. Use a calculator to estimate all solutions to the equation that make $y = 0.004$ over the interval [0, 0.02].



The first point of intersection is at about $x = 0.0017$ sec. The other points of intersection are at about $x = 0.0083$ sec and $x = 0.015$ sec.