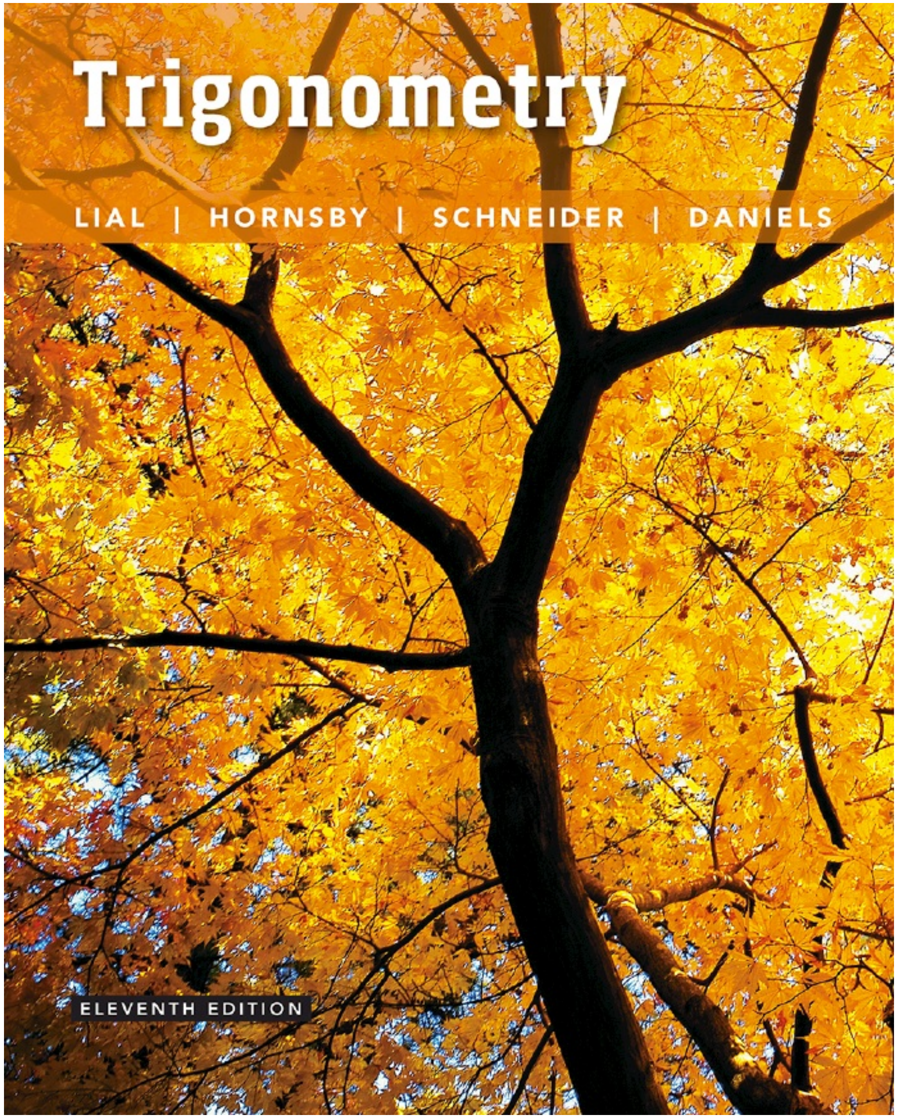
7

Applications of Trigonometry and Vectors



7.1 Oblique Triangles and The Law of Sines

Congruency and Oblique Triangles ▪ Derivation of the Law of Sines ▪ Solutions of SAA and ASA Triangles (Case 1) the Law of Sines ▪ Area of a Triangle

Congruence Axioms

Side-Angle-Side (SAS)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

Angle-Side-Angle (ASA)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

Congruence Axioms

Side-Side-Side (SSS)

If three sides of one triangle are equal, respectively, to three sides of a second triangle, then the triangles are congruent.

Oblique Triangles

Oblique triangle A triangle that is not a right triangle

The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known.

Data Required for Solving Oblique Triangles

Case 1 One side and two angles are known (SAA or ASA).

Case 2 Two sides and one angle not included between the two sides are known (SSA). This case may lead to more than one triangle.

Case 3 Two sides and the angle included between the two sides are known (SAS).

Case 4 Three sides are known (SSS).

▶ Note

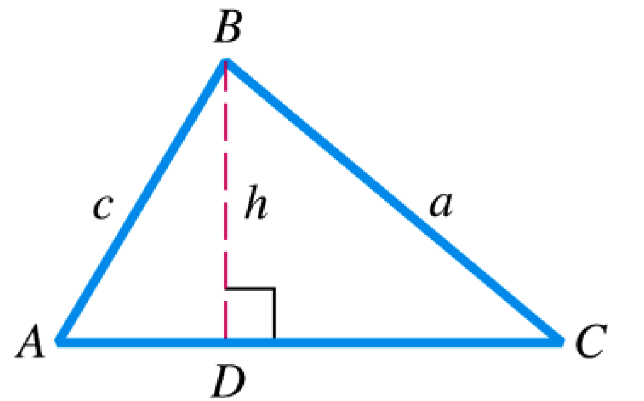
If we know three angles of a triangle, we cannot find unique side lengths since* AAA *assures us only similarity, not congruence.

Derivation of the Law of Sines

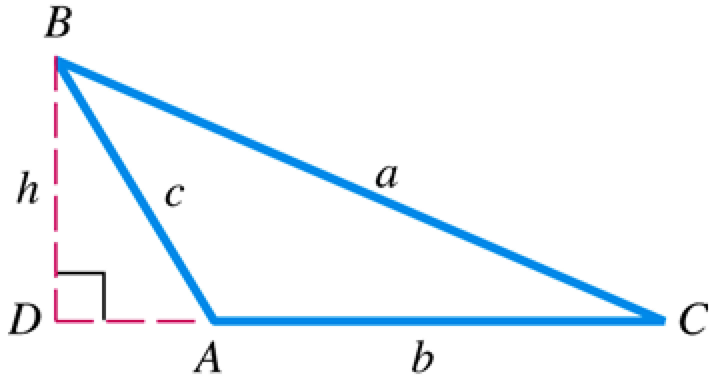
Start with an oblique triangle, either acute or obtuse.

Let *h* be the length of the perpendicular from vertex *B* to side *AC* (or its extension).

Then *c* is the hypotenuse of right triangle *ABD*, and *a* is the hypotenuse of right triangle *BDC.*



Acute triangle *ABC*



Obtuse triangle *ABC*

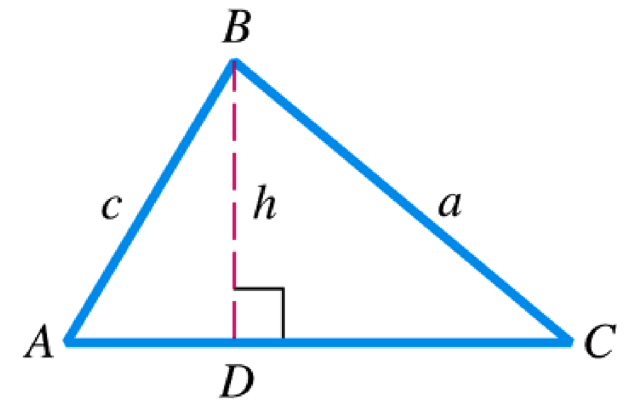
Derivation of the Law of Sines

In triangle *ADB*,

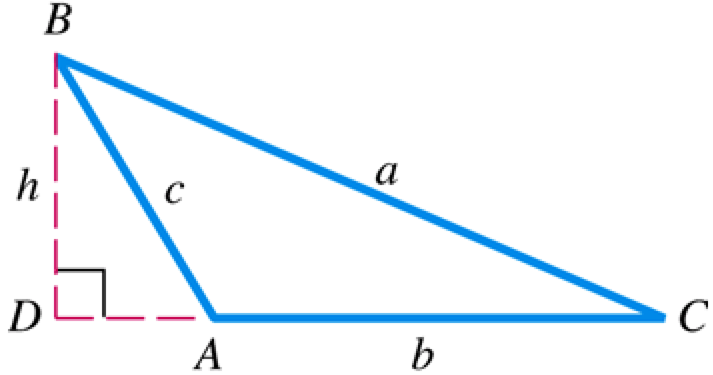
$$\sin A = \frac{h}{c} \text{ or } h = c\sin A$$

In triangle *BDC*,

$$\sin C = \frac{h}{a} \text{ or } h = a\sin C$$



Acute triangle ABC



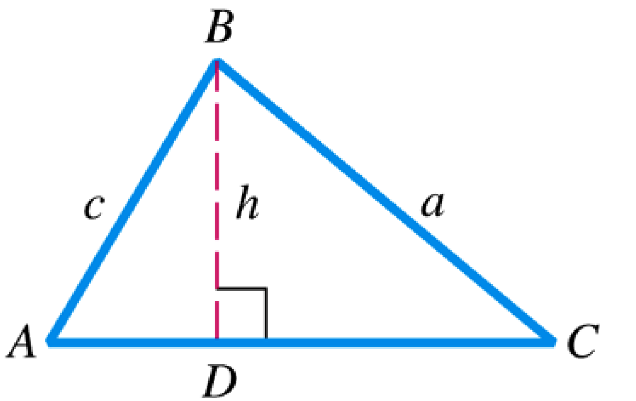
Obtuse triangle ABC

Derivation of the Law of Sines

Since $h = c \sin A$ and $h = a \sin C$,

$$a \sin C = c \sin A$$

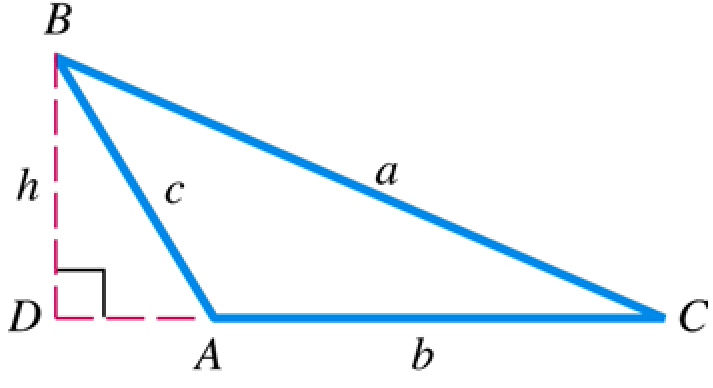
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$



Acute triangle ABC

Similarly, it can be shown that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Obtuse triangle ABC

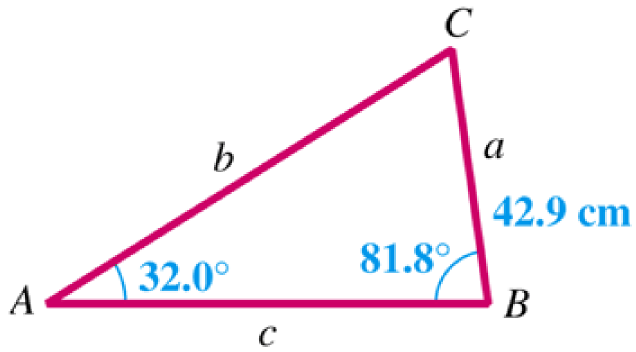
Law of Sines

In any triangle *ABC*, with sides *a*, *b*, and *c*,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 1 APPLYING THE LAW OF SINES (SAA)

Solve triangle *ABC* if $A = 32.0°$, $C = 81.8°$, and $a = 42.9$ cm.



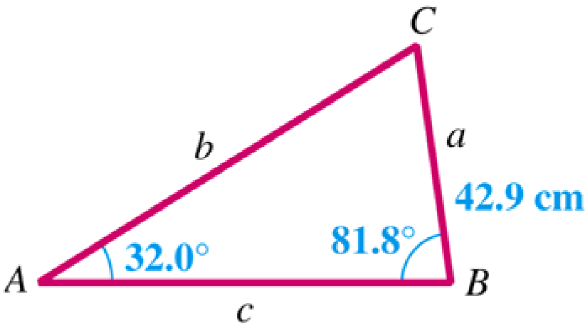
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Law of sines

$$\frac{42.9}{\sin 32.0°} = \frac{b}{\sin 81.8°}$$

$$b = \frac{42.9 \sin 81.8°}{\sin 32.0°}$$

$$\approx 80.1 \text{ cm}$$



$$A + B + C = 180°$$

$$C = 180° - A - B$$

$$C = 180° - 32.0° - 81.8° = 66.2°$$

Use the Law of Sines to find *c*.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{42.9}{\sin 32.0°} = \frac{c}{\sin 66.2°}$$

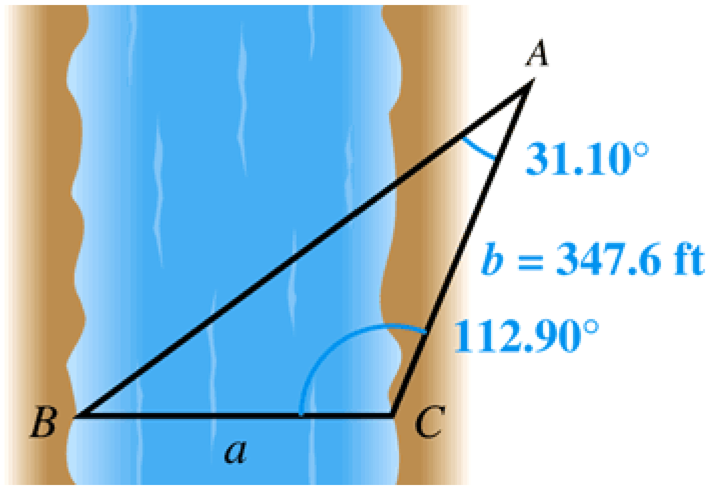
$$c = \frac{42.9 \sin 66.2°}{\sin 32.0°} \approx 74.1 \text{ cm}$$

Caution

Whenever possible, use the given values in solving triangles, rather than values obtained in intermediate steps, to avoid possible rounding errors.

Example 2 APPLYING THE LAW OF SINES (ASA)

Kurt Daniels wishes to measure the distance across the Gasconde River. He determines that $C = 112.90°$, $A = 31.10°$, and $b = 347.6$ ft. Find the distance a across the river.



First find the measure of angle B.

$B = 180° - A - C = 180° - 31.10° - 112.90° = 36.00°$

Example 2 APPLYING THE LAW OF SINES (ASA) (continued)

Now use the Law of Sines to find the length of side *a*.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 31.10^\circ} = \frac{347.6}{\sin 36.00^\circ}$$

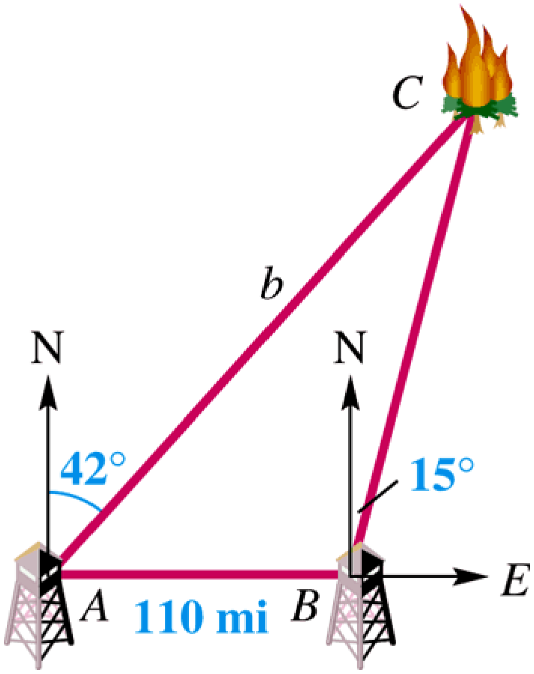
$$a = \frac{347.6 \sin 31.10^\circ}{\sin 36.00^\circ}$$

$$\approx 305.5 \text{ ft}$$

The distance across the river is about 305.5 feet.

Example 3 APPLYING THE LAW OF SINES (ASA)

Two ranger stations are on an east-west line 110 mi apart. A forest fire is located on a bearing N 42° E from the western station at *A* and a bearing of N 15° E from the eastern station at *B*. To the nearest ten miles, how far is the fire from the western station?



First, find the measures of the angles in the triangle.

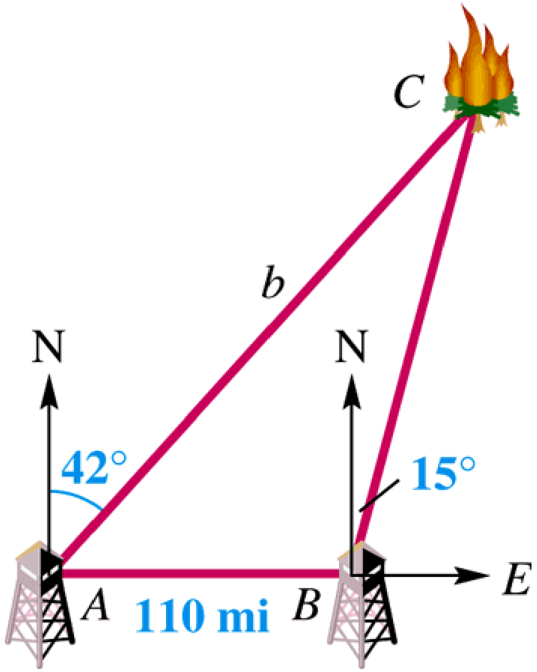
$$m\angle BAC = 90^\circ - 42^\circ = 48^\circ$$

$$m\angle ABC = 90^\circ + 15^\circ = 105^\circ$$

$$m\angle C = 180^\circ - 105^\circ - 48^\circ = 27^\circ$$

▶ Example 3 APPLYING THE LAW OF SINES (ASA) (continued)

Now use the Law of Sines to find *b*.



$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 105°} = \frac{110}{\sin 27°}$$

$$b = \frac{110 \sin 105°}{\sin 27°}$$

$$b \approx 230 \text{ mi}$$

Use a calculator and give two significant digits.

Area of a Triangle (SAS)

In any triangle *ABC*, the area $\mathcal{A}$ is given by the following formulas:

$$\mathcal{A} = \frac{1}{2}bc\sin A \qquad \mathcal{A} = \frac{1}{2}ab\sin C$$

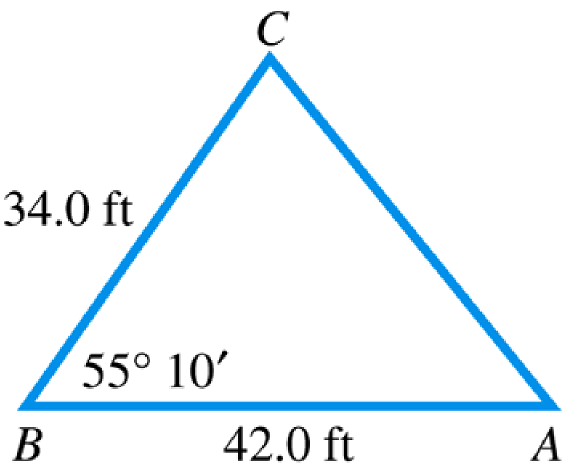
$$\mathcal{A} = \frac{1}{2}ac\sin B$$

Note

If the included angle measures 90°, its sine is 1, and the formula becomes the familiar $\mathcal{A} = \frac{1}{2}bh$.

Example 4 **FINDING THE AREA OF A TRIANGLE (SAS)**

Find the area of triangle *ABC*.



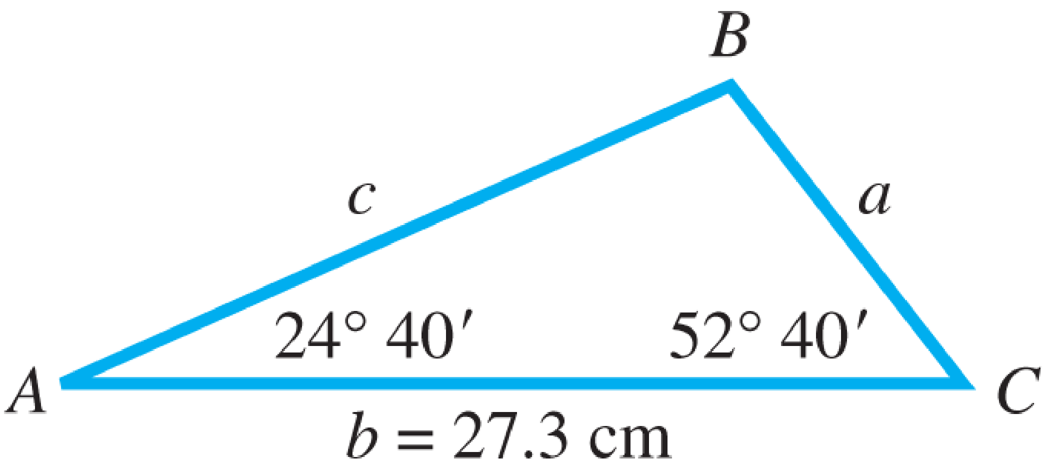
$$\mathcal{A} = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(34.0)(42.0)\sin 55°10'$$

$$\approx 586 \text{ ft}^2$$

Example 5 FINDING THE AREA OF A TRIANGLE (ASA)

Find the area of triangle *ABC*.

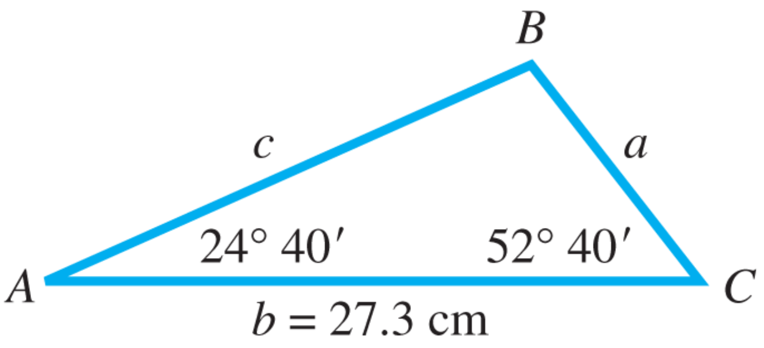


Before the area formula can be used, we must find either *a* or *c*.

$$B = 180° - 24°40' - 52°40' = 102°40'$$

Example 5 FINDING THE AREA OF A TRIANGLE (ASA) (continued)

$$\frac{a}{\sin a} = \frac{b}{\sin B}$$



$$\frac{a}{\sin 24°40'} = \frac{27.3}{\sin 102°40'}$$

$$a = \frac{27.3 \sin 24°40'}{\sin 102°40'} \approx 11.7 \text{ cm}$$

Now find the area.

$$\mathcal{A} = \frac{1}{2} ab \sin C = \frac{1}{2}(11.7)(27.3)\sin 52°40'$$

$$\approx 127 \text{ cm}^2$$