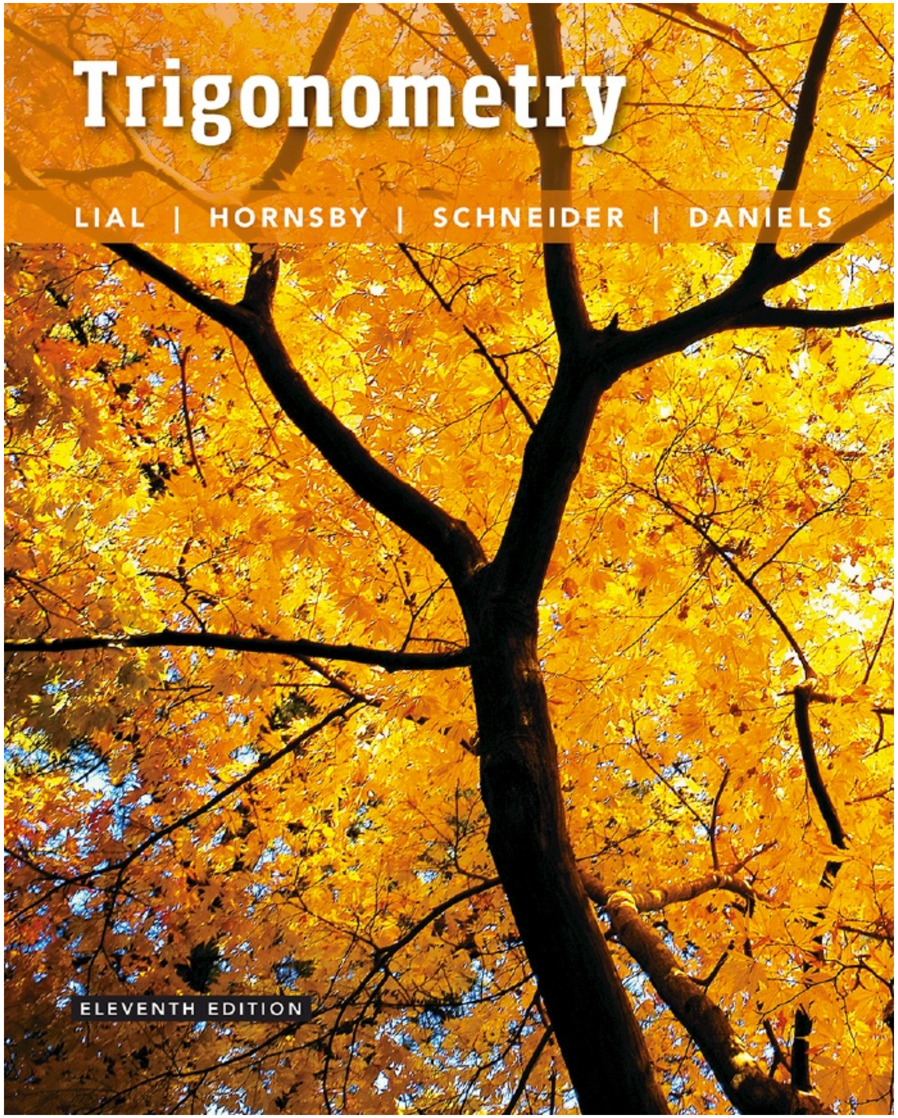
7

Applications of Trigonometry and Vectors



7.3 The Law of Cosines

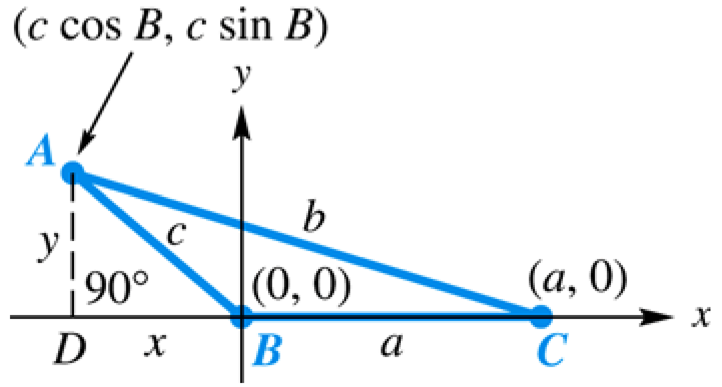
Derivation of the Law of Cosines ▪ Solutions of SAS and SSS Triangles (Cases 3 and 4) ▪ Heron's Formula for the Area of a Triangle ▪ Derivation of Heron's Formula

Triangle Side Length Restriction

In any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

Derivation of the Law of Cosines

Let *ABC* be any oblique triangle located on a coordinate system as shown.



The coordinates of *A* are (*x*, *y*). For angle *B*,

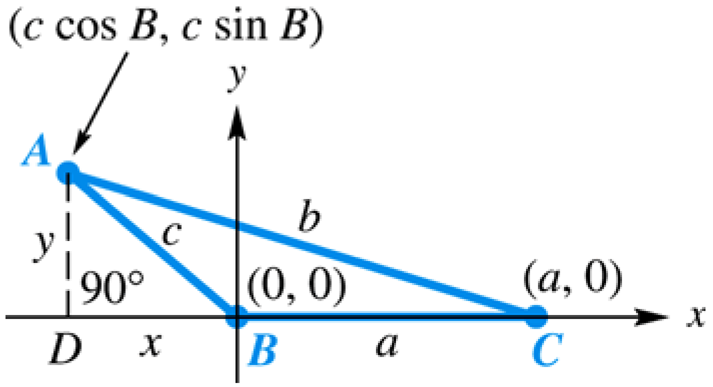
$$\sin B = \frac{y}{c} \Rightarrow y = c\sin B \text{ and } \cos B = \frac{x}{c} \Rightarrow x = c\cos B.$$

Thus, the coordinates of *A* become (c cos B, c sin B).

Derivation of the Law of Cosines (continued)

The coordinates of C are $(a, 0)$ and the length of AC is b.

Using the distance formula, we have



$$b=\sqrt{(c\cos B-a)^2+(c\sin B-0)^2}$$

$$\begin{aligned} b^2 &= c^2\cos^2 B-2ac\cos B+a^2+c^2\sin^2 B \\ &= a^2+c^2\left(\sin^2 B+\cos^2 B\right)-2ac\cos B \\ &= a^2+c^2-2ac\cos B \end{aligned}$$

Square both sides and expand.

$\sin^2 B+\cos^2 B=1$

Law of Cosines

In any triangle *ABC*, with sides *a*, *b*, and *c*,

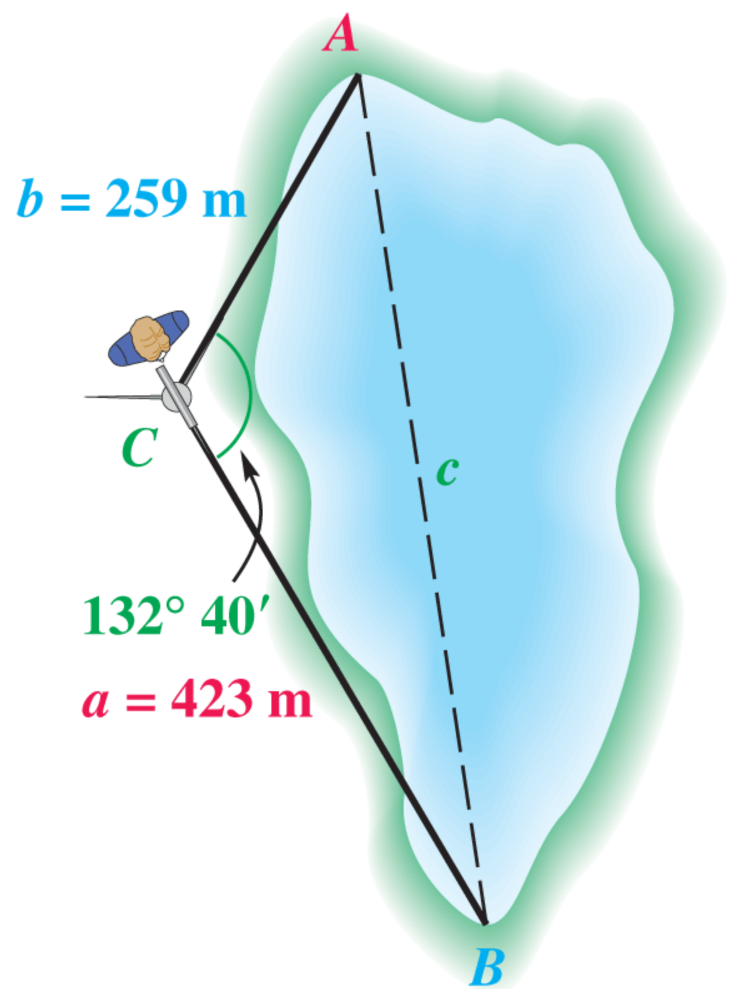
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note

If $C = 90°$ in the third form of the law of cosines, then $\cos C = 0$, and the formula becomes $c^2 = a^2 + b^2$, the Pythagorean theorem.

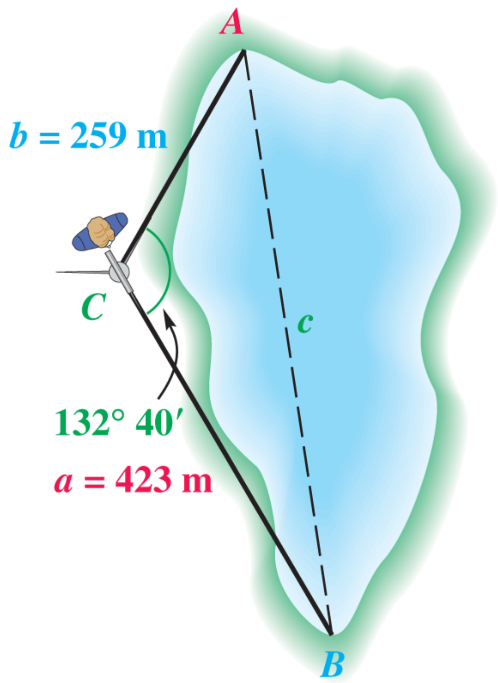
Example 1 APPLYING THE LAW OF COSINES (SAS)

A surveyor wishes to find the distance between two inaccessible points *A* and *B* on opposite sides of a lake. While standing at point *C*, she finds that $b = 259$ m, $a = 423$ m, and angle *ACB* measures 132°40′. Find the distance *c*.



Example 1 APPLYING THE LAW OF COSINES (SAS) (continued)

Use the law of cosines because we know the lengths of two sides of the triangle and the measure of the included angle.



$$c^2 = a^2 + b^2 = 2ab\cos C$$

$$c^2 = 423^2 + 259^2 = 2(423)(259)\cos 132°40'$$

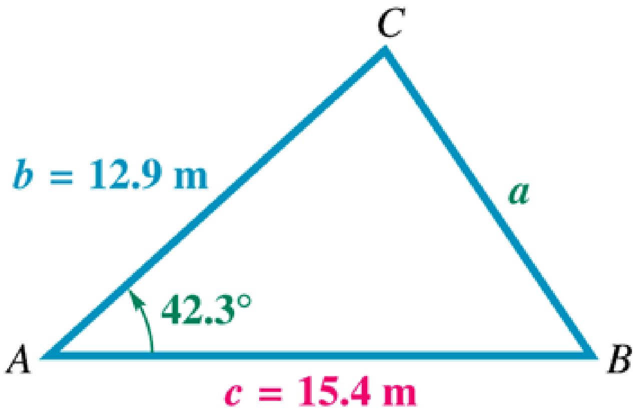
$$c^2 \approx 394{,}510.6$$

$$c \approx 628$$

The distance between the two points is about 628 m.

APPLYING THE LAW OF COSINES (SAS)

Solve triangle *ABC* if $A = 42.3°$, $b = 12.9$ m, and $c = 15.4$ m.



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = 12.9^2 + 15.4^2 - 2(12.9)(15.4)\cos 42.3°$$

$$a \approx 10.47 \text{ m}$$

B < *C* since it is opposite the shorter of the two sides *b* and *c*. Therefore, *B* cannot be obtuse.

Example 2 APPLYING THE LAW OF COSINES (SAS) (continued)

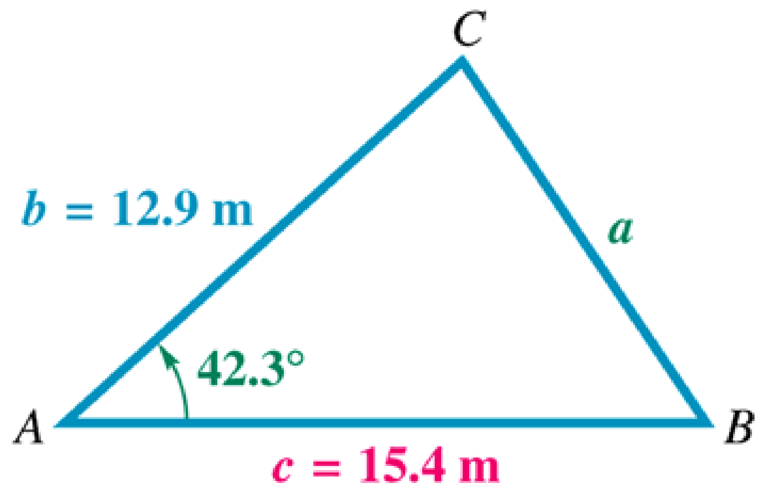
Use the law of sines to find the measure of another angle.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 42.3^\circ}{10.47} = \frac{\sin B}{12.9}$$

$$\sin B = \frac{12.9 \sin 42.3^\circ}{10.47}$$

$$B \approx 56.0^\circ$$



Now find the measure of the third angle.

$$C = 180^\circ - 42.3^\circ - 56.0^\circ = 81.7^\circ$$

Caution

Had we used the law of sines to find *C* rather than *B* in **Example 2**, we would not have known whether *C* was equal to 81.7° or its supplement, 98.3°.

Example 3 APPLYING THE LAW OF COSINES (SSS)

Solve triangle *ABC* if a = 9.47 ft, b = 15.9 ft, and c = 21.1 ft.

Use the law of cosines to find the measure of the largest angle, *C*. If cos *C* < 0, angle *C* is obtuse.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Solve for cosC.

$$\cos C = \frac{9.47^2 + 15.9^2 - 21.1^2}{2(9.47)(15.9)} \approx -.34109402$$

$$C \approx 109.9^\circ$$

Example 3 APPLYING THE LAW OF COSINES (SSS) (continued)

Use the law of sines to find the measure of angle *B*.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin 109.9^\circ}{21.1} = \frac{\sin B}{15.9}$$

$$\sin B = \frac{15.9 \sin 109.9^\circ}{21.1}$$

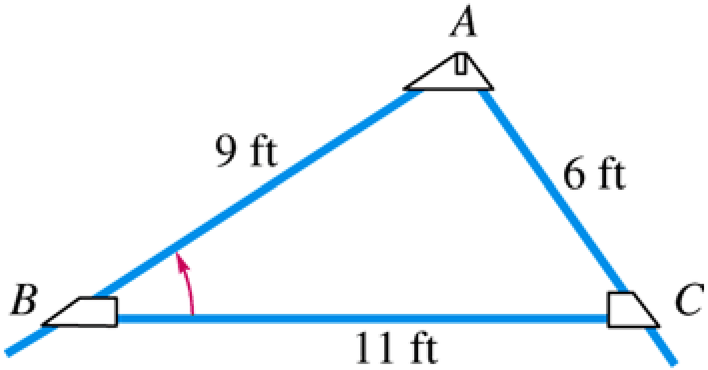
$$B \approx 45.1^\circ$$

Now find the measure of angle *A*.

$$A = 180^\circ - 45.1^\circ - 109.9^\circ = 25.0^\circ$$

Example 4 DESIGNING A ROOF TRUSS (SSS)

Find angle *B* to the nearest degree for the truss shown in the figure.



$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{11^2 + 9^2 - 6^2}{2(11)(9)}$$

$$B \approx 33^\circ$$

Four possible cases can occur when solving an oblique triangle.

Oblique Triangle	Suggested Procedure for Solving
Case 1: One side and two angles are known. **(SAA or ASA)**	***Step 1*** Find the remaining angle using the angle sum formula ($A + B + C = 180°$). ***Step 2*** Find the remaining sides using the law of sines.
Case 2: Two sides and one angle (not included between the two sides) are known. **(SSA)**	*This is the ambiguous case; there may be no triangle, one triangle, or two triangles.* ***Step 1*** Find an angle using the law of sines. ***Step 2*** Find the remaining angle using the angle sum formula. ***Step 3*** Find the remaining side using the law of sines. *If two triangles exist, repeat Steps 2 and 3.*

Oblique Triangle	Suggested Procedure for Solving
Case 3: Two sides and the included angle are known. **(SAS)**	***Step 1*** Find the third side using the law of cosines. ***Step 2*** Find the smaller of the two remaining angles using the law of sines. ***Step 3*** Find the remaining angle using the angle sum formula.
Case 4: Three sides are known. **(SSS)**	***Step 1*** Find the largest angle using the law of cosines. ***Step 2*** Find either remaining angle using the law of sines. ***Step 3*** Find the remaining angle using the angle sum formula.

Heron's Area Formula (SSS)

If a triangle has sides of lengths *a*, *b*, and *c*, with **semiperimeter**

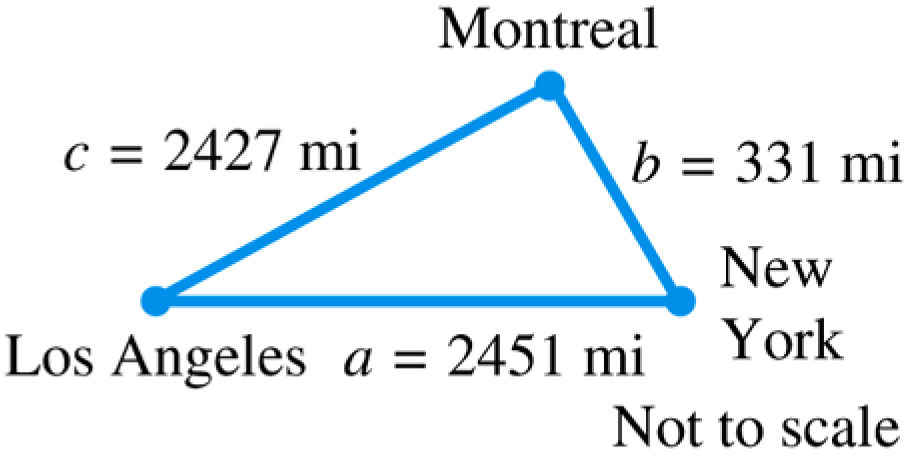
$$s = \frac{1}{2}(a + b + c)$$

then the area of the triangle is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

Example 5 USING HERON'S FORMULA TO FIND AN AREA (SSS)

The distance "as the crow flies" from Los Angeles to New York is 2451 mi, from New York to Montreal is 331 mi, and from Montreal to Los Angeles is 2427 mi. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)



Example 5 USING HERON'S FORMULA TO FIND AN AREA (SSS) (continued)

The semiperimeter s is

$$s = \frac{1}{2}(2451 + 331 + 2427) = 2604.5$$

Using Heron's formula, the area $\mathcal{A}$ is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{2604.5(2604.5-2451)\cdot(2604.5-331)(2604.5-2427)}$$

$$\approx 401{,}700 \text{ mi}^2$$

Derivation of Heron's Formula

Let triangle *ABC* have sides of length *a*, *b*, and *c*. Apply the law of cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The perimeter of the triangle is $a + b + c$, so half the perimeter (the semiperimeter) is given by the formula in equation (2) shown next.

Derivation of Heron's Formula (continued)

$$s = \frac{1}{2}(a + b + c) \quad (2)$$

$$2s = a + b + c \quad (3)$$

$$b + c - a = 2s - 2a$$

$$b + c - a = 2(s - a)$$

Subtract 2*b* and 2*c* in a similar way in equation (3) to obtain equations (5) and (6).

$$a - b + c = 2(s - b) \quad (5)$$

$$a + b - c = 2(s - c) \quad (6)$$

Derivation of Heron's Formula (continued)

Now we obtain an expression for 1 – cos *A.*

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc + a^2 - b^2 - c^2}{2bc}$$

$$= \frac{a^2 - \left(b^2 - 2bc + c^2\right)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

Derivation of Heron's Formula (continued)

$$\frac{a^2-(b-c)^2}{2bc}=\frac{[a-(b-c)][a+(b-c)]}{2bc}$$

$$=\frac{(a-b+c)(a+b-c)}{2bc}$$

$$=\frac{2(s-b)\bullet 2(s-c)}{2bc}$$

$$1-\cos A=\frac{2(s-b)(s-c)}{bc}$$

Similarly, it can be shown that

$$1+\cos A=\frac{2(s-a)}{bc}$$

Derivation of Heron’s Formula (continued)

Recall the double-angle identities for cos 2θ.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1$$

$$1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$$

$$\frac{2s(s-a)}{bc} = 2\cos^2\left(\frac{A}{2}\right)$$

$$\frac{s(s-a)}{bc} = \cos^2\left(\frac{A}{2}\right)$$

$$\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}} \quad (9)$$

Derivation of Heron's Formula (continued)

Recall the double-angle identities for cos 2θ.

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$$

$$1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$$

$$\frac{2(s-b)(s-c)}{bc} = 2\sin^2\left(\frac{A}{2}\right)$$

$$\frac{(s-b)(s-c)}{bc} = \sin^2\left(\frac{A}{2}\right)$$

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad (10)$$

Derivation of Heron's Formula (continued)

The area of triangle *ABC* can be expressed as follows.

$$\mathcal{A} = \frac{1}{2} bc \sin A$$

$$2\mathcal{A} = bc \sin A$$

$$\frac{2\mathcal{A}}{bc} = \sin A \quad (11)$$

Derivation of Heron's Formula (continued)

Recall the double-angle identity for sin 2θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$\frac{2\mathcal{A}}{bc} = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$\frac{2\mathcal{A}}{bc} = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \bullet \sqrt{\frac{s(s-a)}{bc}}$$

Derivation of Heron's Formula (continued)

$$\frac{2\mathcal{A}}{bc} = 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}}$$

$$\frac{2\mathcal{A}}{bc} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$ Heron's formula