5.5. Double Angle Identities and Product-to-Sum and Sum-to-Product Identities

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Last time:

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

Obj 1: Double Angle Identies

$$\cos(2A) = \cos(A+A) = \cos A \cdot \cos A - \sin A \cdot \sin A$$

$$= \cos^2 A - \sin^2 A$$

$$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$$

$$\cos(2A) = \underbrace{\boxed{\cos^2 A}}_{(1 - \sin^2 A)} - \sin^2 A$$

$$(1 - \sin^2 A) - \sin^2 A$$

$$\boxed{\cos(2A) = 1 - 2\sin^2 A}$$

$$\cos(2A) = \cos^2 A - \underbrace{\boxed{\sin^2 A}}_{(1-\cos^2 A)}$$

$$= \cos^2 A - (1 - \cos^2 A)$$

$$\boxed{\cos(2A) = 2\cos^2 A - 1}$$

Double-Angle Identity for Sine.

$$\sin(2A) = \sin(A + A) = \sin A \cos A + \cos A \sin A$$

$$\boxed{\sin(2A) = 2 \sin A \cos A}$$

Double-angle Identity for tangent

$$\tan(2A) = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A}$$

$$\boxed{\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}}$$

All the Double-Angle Identities.

$$\cos(2A) = \cos^2 A - \sin^2 A; \quad \cos(2A) = 1 - 2\sin^2 A$$

$$\cos(2A) = 2\cos^2 A - 1$$

$$\sin(2A) = 2\sin A \cos A$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

Ex. 1. Given that $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$

Q: Find $\cos(2\theta)$, $\sin(2\theta)$, $\tan(2\theta)$.

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$= 2\cdot\left(\frac{9}{25}\right) - 1 = \frac{18}{25} - 1 = \boxed{-\frac{7}{25}}$$

$$\sin(2\theta) = 2\boxed{\sin\theta}\boxed{\cos\theta} = 2\cdot\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = \boxed{\frac{-24}{25}}$$

? $\qquad \frac{3}{5}$

$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1 \rightarrow \sin^2\theta + \frac{9}{25} = 1$$

$$\sin^2\theta = 1 - \frac{9}{25} \longrightarrow \sin^2\theta = \frac{16}{25}$$

$\longrightarrow \sin\theta = \pm\frac{4}{5}$. Since we are given $\sin\theta < 0$,

$$\sin\theta = -\frac{4}{5}$$

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$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{\frac{-24}{25}}{\frac{-7}{25}} = \frac{24}{\cancel{25}} \cdot \frac{\cancel{25}}{7} = \frac{24}{7}$$

$$\tan(2\theta) = \frac{24}{7}$$

Ex. 2. Given that $\cos(2\theta) = \frac{4}{5}$ and $90° < \theta < 180°$

Q: Find $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$.

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\frac{4}{5} = 2\cos^2\theta - 1$$

$$\rightarrow \frac{9}{5} = 2\cos^2\theta \longrightarrow \cos^2\theta = \frac{9}{10}$$

$$\rightarrow \cos\theta = \pm\sqrt{\frac{9}{10}}.$$ Since θ is in Quadrant II,

$$\cos\theta = -\sqrt{\frac{9}{10}} = -\frac{3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \boxed{-\frac{3\sqrt{10}}{10}}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \left(-\frac{3\sqrt{10}}{10}\right)^2 = 1$$

$$\rightarrow \sin^2\theta + \frac{9}{10} = 1 \rightarrow \sin^2\theta = \frac{1}{10}$$

$$\rightarrow \sin\theta = \pm\sqrt{\frac{1}{10}} \rightarrow \sin\theta = \sqrt{\frac{1}{10}} \quad (\theta \text{ is in II})$$

$$\boxed{\sin\theta = \frac{\sqrt{10}}{10}}$$

Easy to find $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$ now.

Ex. 3. Verify this identity:

$$\cot(\theta)\cdot\sin(2\theta) = 1 + \cos(2\theta).$$

$$\text{LHS} = \frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{\sin(2\theta)}{1} = \frac{\cos(\theta)\cdot\sin(2\theta)}{\sin(\theta)}$$

$$= \frac{\cos(\theta)\cdot 2\sin(\theta)\cdot\cos(\theta)}{\sin(\theta)} = \frac{2\cdot\cos^2(\theta)\cdot\cancel{\sin(\theta)}}{\cancel{\sin(\theta)}}$$

$$LHS = 2\cos^2(\theta)$$

$$RHS = 1 + \underbrace{\cos(2\theta)} = \cancel{1} + \underbrace{2\cos^2(\theta) \cancel{-1}}$$

$$= 2\cos^2(\theta)$$

Done!

Ex. 4. Find the exact value of

$$\sin 15^\circ \cos 15^\circ$$

$$\boxed{\sin(2 \cdot 15^\circ)} = 2\boxed{\sin 15^\circ \cos 15^\circ}$$

?

$$= \sin(30^\circ) = \frac{1}{2}$$

$$\rightarrow \frac{1}{2} = 2\boxed{\sin 15^\circ \cos 15^\circ}$$

$$\rightarrow \boxed{\sin 15^\circ \cos 15^\circ = \frac{1}{4}}$$

Ex.5 Find the exact value of

$$\cos^2(22.5^\circ) - \sin^2(22.5^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}.$$

Ex. 6. Write $\sin(3x)$ in terms of $\sin(x)$.

$$\sin(3x) = \sin(2x + x)$$

$$= \underbrace{\sin(2x)}_{\downarrow} \cdot \cos x + \cos(2x) \cdot \sin x$$

$$= 2\sin x \cos x \cdot \cos x + (1 - 2\sin^2 x) \cdot \sin x$$

$$= 2\sin x \cdot \underbrace{\cos^2 x}_{\downarrow} + (1 - 2\sin^2 x) \cdot \sin x$$

$$= 2\sin x \cdot (1 - \sin^2 x) + (1 - 2\sin^2 x) \cdot \sin x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$\boxed{\sin(3x) = 3\sin x - 4\sin^3 x}$$

Obj 2: Product-to-Sum and Sum-to-Product Formula.

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

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