5.6-Half-Angle Identities

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Obj 1: Develop the half-angle identities

Half-Angle Identity for Sine

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\cos(2\theta) - 1 = -2\sin^2\theta$$

$$\sin^2\theta = \frac{\cos(2\theta) - 1}{-2}$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

Let $A = 2\theta$. Then $\theta = \frac{A}{2}$

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos(A)}{2}}$$

* Half - Angle Identity for Cosine.

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) + 1 = 2\cos^2\theta$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\cos\theta = \pm\sqrt{\frac{1 + \cos(2\theta)}{2}}$$

Let $A = 2\theta$. Then $\theta = \frac{A}{2}$.

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos(A)}{2}}$$

* Half - Angle Identity for tangent.

$$\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos(A)}{2}}}{\pm\sqrt{\frac{1+\cos(A)}{2}}}$$

$$= \pm\sqrt{\frac{\frac{1-\cos(A)}{2}}{\frac{1+\cos(A)}{2}}}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}}$$

$$* \quad \tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right) \cdot 2\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right) \cdot 2\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A}{2}\right) \cdot \cos\left(\frac{A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)}$$

$$= \frac{\sin(A)}{\cos(A) + 1}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{\cos(A) + 1}}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \frac{1 - \cos(A)}{\sin(A)}}$$

Summary of the half-angle identities.

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{\cos(A)+1}; \quad \tan\left(\frac{A}{2}\right) = \frac{1-\cos(A)}{\sin(A)}$$

Ex. 1. Find the exact value of $\sin 15^\circ$, $\cos 15^\circ$, $\tan 15^\circ$. Simplify the answers as much as possible

Ex. 2 Find the exact value of $\tan(22.5^\circ)$

Sol: $\sin 15^\circ = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

$$\cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$\tan 15^\circ = \frac{1-\cos(30^\circ)}{\sin(30^\circ)} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\tan 15^\circ = \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$$

$$\tan 15^\circ = 2-\sqrt{3}$$

Ex. 2

$$\tan(22.5^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)+1}$$

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+1} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}+2}{2}}$$

$$= \frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{2}+2}$$

$$= \frac{\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}(2-\sqrt{2})}{4-2}$$

$$= \frac{\sqrt{2}(2-\sqrt{2})}{2} = \frac{\sqrt{2}\cdot\sqrt{2}(\sqrt{2}-1)}{2}$$

$$\tan(22.5^\circ) = \sqrt{2}-1$$

Ex. Given that $\cos\theta = \frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$

Q: Find $\cos\left(\frac{\theta}{2}\right)$; $\sin\left(\frac{\theta}{2}\right)$; $\tan\left(\frac{\theta}{2}\right)$