5.6-Half-Angle Identities

Monday, November 13, 2017 9:18 AM

Last time: Double-Angle Identities

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$= 1 - 2\sin^2(A)$$
$$= 2\cos^2(A) - 1$$

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

Today:

Obj 1: Develop the half-angle identities

Half-angle identity for Sine.

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$-2\sin^2\theta = \cos(2\theta) - 1$$

$$\sin^2\theta = \frac{\cos(2\theta) - 1}{-2}$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}$$

Let $2\theta = A$.

Then $\theta = \frac{A}{2}$.

$$\boxed{\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}}$$

Half-Angle Identity for Cosine

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$2\cos^2\theta = \cos(2\theta) + 1$$

$$\cos^2\theta = \frac{\cos(2\theta)+1}{2}$$

$$\cos(\theta) = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

Let $A = 2\theta$. Then $\theta = \frac{A}{2}$.

The last identity becomes

$$\boxed{\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}}$$

Half-Angle Identity for tangent

$$\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \frac{\pm\sqrt{\frac{1-\cos(A)}{2}}}{\pm\sqrt{\frac{1+\cos(A)}{2}}}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)\cdot 2\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)\cdot 2\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A}{2}\right)\cdot\cos\left(\frac{A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)} = \frac{\sin(A)}{\cos(A)+1}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{\cos(A)+1}}$$

Summary of Half-Angle Identities:

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}} \quad ; \quad \cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}} \quad ; \quad \tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{\cos(A)+1}$$

$$\tan\left(\frac{A}{2}\right) = \frac{1-\cos(A)}{\sin(A)}$$

Ex. 1. Find the exact value of $\sin 15^\circ$, $\cos 15^\circ$ and $\tan 15^\circ$. Simplify your result as much as possible.

$$\sin 15^\circ = \sqrt{\frac{1-\cos 30^\circ}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$\boxed{\sin 15^\circ = \frac{\sqrt{2-\sqrt{3}}}{2}}$$

$$\cos 15^\circ = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$\boxed{\cos 15^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}}$$

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{2-\sqrt{3}}}{2}}{\frac{\sqrt{2+\sqrt{3}}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}$$

Ex. Find the exact value of $\cos(22.5°)$ and $\tan(22.5°)$

$$\cos(22.5°) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\tan(22.5°) = \frac{\sin(45°)}{\cos(45°)+1} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+1}$$

$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}+2}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}(2-\sqrt{2})}{4-2} = \frac{\sqrt{2}(2-\sqrt{2})}{2}$$

$$= \frac{2-\sqrt{2}}{\sqrt{2}} = \frac{2}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

Ex. Given that $\cos\theta = \frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$.

Find $\cos\left(\frac{\theta}{2}\right)$; $\sin\left(\frac{\theta}{2}\right)$; $\tan\left(\frac{\theta}{2}\right)$

Find $\cos\left(\frac{\theta}{2}\right)$

θ is in quadrant IV

$\Rightarrow \frac{\theta}{2}$ is in quadrant II.

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}}$$

$$= -\sqrt{\frac{1+\frac{2}{3}}{2}} = -\sqrt{\frac{\frac{5}{3}}{2}} = -\sqrt{\frac{5}{6}}$$

$$= -\frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \boxed{-\frac{\sqrt{30}}{6}}$$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{2}{3}}{2}}$$

$$= \sqrt{\frac{\frac{1}{3}}{2}} = \sqrt{\frac{1}{6}} = \boxed{\frac{\sqrt{6}}{6}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}} = -\frac{\sqrt{6}}{\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}}$$

$$= \frac{-\sqrt{180}}{30} = \frac{-\sqrt{36 \cdot 5}}{30} = \frac{-6\sqrt{5}}{30} = \frac{-\sqrt{5}}{5}.$$