# 6.3. The Dual Problem

Tuesday, October 31, 2017 8:34 AM

**Goal:** Solve the minimization problem with constraints of the form $\geq$

**Recall:** Last time, we solved the maximization problem with constraints of the form $\leq$

In particular, $\boxed{\text{Maximize}}$ $P = 5x + 10y$

Subject to: $8x + 8y \boxed{\leq} 160$

$4x + 12y \boxed{\leq} 180$

Today, the problem is this:

$\boxed{\text{Minimize}}$ $1C = \boxed{16}x_1 + \boxed{9}x_2 + \boxed{21}x_3$

$$\left.\begin{aligned} \text{Subject to: } 1x_1 + 1x_2 + \boxed{3}x_3 &\boxed{\geq} \boxed{12} \\ \boxed{2}x_1 + 1x_2 + 1x_3 &\boxed{\geq} \boxed{16} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}\right\} \text{Constraints}$$

**Key idea for solving Minimization problem:** to translate this to the maximization problem with constraints of the form $\leq$ and apply the simplex method from last time.

**Here are the Steps:**

**Step 1:** Write down the initial matrix for the problem.
((Coefficients for objective function will be in bottom row)

3-by-4

$$\text{Initial Matrix } A = \begin{pmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \end{pmatrix}$$

**Step 2:** Find the transpose of the initial matrix.

4-by-3

$$A^T = \begin{pmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{pmatrix}$$

**Step 3:** Use the transpose of the original matrix to rewrite the minimization problem into a maximization problem.

$\longrightarrow$ This is called forming the dual problem.

$$\begin{pmatrix} \boxed{1 \quad 2 \quad 16} \\ \boxed{1 \quad 1 \quad 9} \\ \boxed{3 \quad 1 \quad 21} \\ \boxed{12 \quad 16 \quad 1} \end{pmatrix}$$

$y_1 + 2y_2 \leq 16$

$y_1 + y_2 \leq 9$

$3y_1 + y_2 \leq 21$

$y_1, y_2 \geq 0$

$$\to \underbrace{P = 12y_1 + 16y_2}_{\text{Objective function} \to \text{maximize}}$$

The dual problem is:

Maximize $P = 12y_1 + 16y_2$

Subject to: $y_1 + 2y_2 \leq 16$

$y_1 + y_2 \leq 9$

$3y_1 + y_2 \leq 21$

$y_1, y_2 \geq 0$.

Step 4: Apply the Simplex Method from last time to solve the dual problem.

Note: Introduce Slack Variables. But now we call them $x_1, x_2, x_3, \ldots$ instead of $s_1, s_2, s_3$ like last time.

* Make a Simplex Tableau.
* Find pivot position
* Do row operation, etc.

$$\begin{aligned}
y_1 + 2y_2 + x_1 &= 16 \\
y_1 + y_2 + x_2 &= 9 \\
3y_1 + y_2 + x_3 &= 21 \\
-12y_1 - 16y_2 + P &= 0
\end{aligned}$$

Pivot column

entering var.

Pivot row

exit. var.

$$\begin{array}{c|c}
 & \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ P \end{array} &
\left(\begin{array}{ccccccc}
1 & \boxed{2} & 1 & 0 & 0 & 0 & 16 \\
1 & \boxed{1} & 0 & 1 & 0 & 0 & 9 \\
3 & \boxed{1} & 0 & 0 & 1 & 0 & 21 \\
-12 & \boxed{-16} & 0 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}$$

$$\xrightarrow{R_1 \leftrightarrow \frac{1}{2}R_1} \qquad \xrightarrow[R_3 \leftrightarrow -R_1 + R_3]{R_2 \leftrightarrow -R_1 + R_2} \qquad \xrightarrow{R_4 \leftrightarrow 16R_1 + R_4}$$

enter

exit

$$\begin{array}{c|c}
 & \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\begin{array}{c} y_2 \\ x_2 \\ x_3 \\ P \end{array} &
\left(\begin{array}{cccccc|c}
\boxed{\frac{1}{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 8 \\
\boxed{\frac{1}{2}} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 1 \\
\boxed{\frac{5}{2}} & 0 & -\frac{1}{2} & 0 & 1 & 0 & 13 \\
\boxed{-4} & 0 & 8 & 0 & 0 & 1 & 128
\end{array}\right)
\end{array}$$

$8/\tfrac{1}{2} = 16$

$1/\tfrac{1}{2} = 2$

$13/\tfrac{5}{2} = 26/5$

$-\frac{1}{2}R_2 + R_1$

$-\frac{5}{2}R_2 + R_3$

$4R_2 + R_4$

Pivot row

pivot column

$$\xrightarrow{R_2 \leftrightarrow 2R_2}$$

$$
\begin{array}{c|ccccccc}
 & y_1 & y_2 & x_1 & x_2 & x_3 & P & \\
\hline
y_2 & 0 & 1 & 1 & -1 & 0 & 0 & 7 \\
y_1 & 1 & 0 & -1 & 2 & 0 & 0 & 2 \\
x_3 & 0 & 0 & 2 & -5 & 1 & 0 & 8 \\
P & 0 & 0 & 4 & 8 & 0 & 1 & 136
\end{array}
$$

Step 5: The Solution of the original minimization problem will be read off from the bottom row of the tableau (instead of the rightmost column like last time)

So, $x_1 = 4$ ; $x_2 = 8$ ; $x_3 = 0$

Minimum cost $C = 136$