6.3. The Dual Problem.

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Goal: Solve the minimization problem with constraints of the form $\geq$

Recall: Last time, we solved the maximization problem with constraints of the form $\leq$.

In particular, $\boxed{\text{Maximize}}$ $P = 5x + 10y$

Subject to: $8x + 8y \boxed{\leq} 160$

$4x + 12y \boxed{\leq} 180$

Today, the problem is this:

$\boxed{\text{Minimize}}$: $C = \boxed{16}x_1 + \boxed{9}x_2 + \boxed{21}x_3$

Subject to:
$$\left.\begin{aligned} 1x_1 + 1x_2 + \boxed{3}x_3 &\boxed{\geq} \boxed{12} \\ \boxed{2}x_1 + 1x_2 + 1x_3 &\boxed{\geq} \boxed{16} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}\right\} \text{Constraints}$$

__Key idea:__ to translate this to the maximization problem subject to constraints of the form $\leq$ and apply the simplex method from last time.

Here are the steps:

__Step 1:__ Write the initial matrix for the problem. (Coefficients for objective function in bottom row)

3 by 4

$$\text{Initial Matrix } A = \begin{pmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \end{pmatrix}$$

__Step 2:__ Find the transpose of the initial matrix

$$A^T = \begin{pmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{pmatrix} \longrightarrow \text{4-by-3 matrix}$$

<u>Step 3:</u> Use the transpose of the original matrix to rewrite the minimization problem into a maximization problem

$\longrightarrow$ This is called forming the Dual Problem

$$\begin{pmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ \boxed{12 \quad 16 \quad 1} \end{pmatrix}$$

$\longrightarrow$ $y_1 + 2y_2 \le 16$

$y_1 + y_2 \le 9$

$3y_1 + y_2 \le 21$

$y_1, y_2 \ge 0$

$\longrightarrow$ $\underbrace{P = 12y_1 + 16y_2}_{\text{objective function}}$ $\rightarrow$ maximize

The dual problem is:

$\boxed{\text{Maximize}}$ $P = 12y_1 + 16y_2$

Subject to: $y_1 + 2y_2 \boxed{\leq} 16$

$y_1 + y_2 \boxed{\leq} 9$

$3y_1 + y_2 \boxed{\leq} 21.$

$y_1, y_2 \geq 0$

Step 4: Apply the Simplex method from last time to solve the dual problem.

* Note: Introduce Slack Variables. But now we call them $x_1, x_2, x_3, \ldots$ instead of s_1, s_2, s_3 like last time
* Make the simplex tableau
* Find pivot position
* Do row operations, etc.

Maximize $\boxed{P = 12y_1 + 16y_2}$

Subject to: $y_1 + 2y_2 \le 16$

$y_1 + y_2 \le 9$

$3y_1 + y_2 \le 21$

$$y_1 + 2y_2 + x_1 = 16$$
$$y_1 + y_2 + x_2 = 9$$
$$3y_1 + y_2 + x_3 = 21$$
$$-12y_1 - 16y_2 + P = 0$$

pivot row

exit. var

pivot column

entering var

$$\begin{array}{c|cccccc|c}
 & y_1 & y_2 & x_1 & x_2 & x_3 & P & \\
\hline
x_1 & 1 & \boxed{2} & 1 & 0 & 0 & 0 & 16 \\
x_2 & 1 & 1 & 0 & 1 & 0 & 0 & 9 \\
x_3 & 3 & 1 & 0 & 0 & 1 & 0 & 21 \\
P & -12 & -16 & 0 & 0 & 0 & 1 & 0
\end{array}$$

$$\xrightarrow{R_1 \leftrightarrow \frac{1}{2}R_1} \quad \xrightarrow[R_2 \leftrightarrow -R_1 + R_2]{R_4 \leftrightarrow 16R_1 + R_4;\ R_3 \leftrightarrow -R_1 + R_3}$$

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We got

Pivot column enter

$$\begin{array}{c|cccccc|c} & y_1 & y_2 & x_1 & x_2 & x_3 & P & \\ y_2 & \frac{1}{2} & 1 & \frac{1}{2} & 5 & 0 & 0 & 8 \\ x_2 & \boxed{\frac{1}{2}} & 0 & -\frac{1}{2} & -4 & 0 & 0 & 1 \\ x_3 & \frac{5}{2} & 0 & -\frac{1}{2} & -5 & 1 & 0 & 13 \\ P & -4 & 0 & 8 & 80 & 0 & 1 & 128 \end{array}$$

Pivot row exit

$$\xrightarrow{R_2 \leftrightarrow 2R_2}$$

$$R_1 \leftrightarrow -\frac{1}{2}R_2 + R_1$$

$$R_3 \leftrightarrow -\frac{5}{2}R_2 + R_3$$

$$R_4 \leftrightarrow 4R_2 + R_4$$

$$
\begin{array}{c}
 \\ y_2 \\ y_1 \\ x_3 \\ P
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P & & \end{array} \\
\left(\begin{array}{cccccc|c}
0 & 1 & 1 & 9 & 0 & 0 & 7 \\
1 & 0 & -1 & -8 & 0 & 0 & 2 \\
0 & 0 & 2 & 15 & 1 & 0 & 8 \\
0 & 0 & 4 & 48 & 0 & 1 & 136
\end{array}\right)
\end{array}
$$

<u>Step 5:</u> The solution of the original minimization problem will be read off from the bottom row of the tableau (instead of the rightmost column like last time)

So, $x_1 = 4$; $x_2 = 48$; $x_3 = 0$

Minimum cost $C = 136$.

HW #2

Slack Variables: x_1, x_2.

$$2y_1 + 3y_2 + x_1 = 7$$

$$9y_1 + 8y_2 + x_2 = 4$$

$$-3y_1 - 2y_2 + P = 0$$

Pivot col

y1 enter

$$\begin{array}{c|c} & \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & P & \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} \boxed{2} & 3 & 1 & 0 & 0 & 7 \\ \boxed{9} & 8 & 0 & 1 & 0 & 4 \\ \boxed{-3} & -2 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{c} 3.5 \\ \frac{4}{9} \\ \ \end{array}$$

Pivot row, exit

$$\xrightarrow{R_2 \leftrightarrow \frac{1}{9} R_2} \qquad \xrightarrow[R_3 \leftrightarrow 3R_2 + R_3]{R_1 \leftrightarrow -2R_2 + R_1}$$

$$
\begin{array}{ccccc|c}
 & & x_1 & x_2 & P & \\
0 & \frac{11}{9} & 1 & -\frac{2}{9} & 0 & \frac{5}{9} \\
1 & \frac{8}{9} & 0 & \frac{1}{9} & 0 & \frac{4}{9} \\
0 & \frac{2}{3} & 0 & \frac{1}{3} & 1 & \frac{4}{3}
\end{array}
$$