4.2. Systems of Linear Equations and Augmented Matrices.

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<u>Goals</u>: (1) Understand matrix terminology

(2) Solve a linear system of 2 equations using the augmented matrix

(3) Identify 3 possible matrix types for a linear system of 2 equations.

A matrix is a rectangular array of numbers.

E.g. $\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}$; $\underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}}$; $\underbrace{\begin{pmatrix} 4 & 2 & 1 & 3 \end{pmatrix}}_{\text{1-by-4 matrix}}$

$\underbrace{\text{2-by-2 matrix}}_{\text{Dimension of the matrix}}$

$\underbrace{\begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}}_{\text{4-by-1 matrix}}$

E.g. of a general 2-by-2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \text{entry of the matrix}$$

Augmented Matrix Associated with a linear system.

E.g.

$$\begin{cases} x + 3y = 5 \\ 2x - y = 3 \end{cases} \longrightarrow \underbrace{\left(\begin{array}{cc|c} \boxed{\begin{matrix} 1 & 3 \\ 2 & -1 \end{matrix}} & \begin{matrix} 5 \\ 3 \end{matrix} \end{array}\right)}_{\text{Augmented Matrix Associated with this syst}}$$

Coefficient Matrix

$$\begin{cases} 4x - y = 5 \\ x + 3y = 8 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 4 & -1 & 5 \\ 1 & 3 & 8 \end{array}\right)$$

E.g.

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 7 \end{array}\right) \longrightarrow \begin{cases} 1 \cdot x + 0 \cdot y = 3 \\ 0 \cdot x + 1 \cdot y = 7 \end{cases} \longrightarrow \begin{cases} x = 3 \\ y = 7 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{cases} x + y = 3 \\ 0x + 0y = 0 \end{cases} \rightarrow \begin{cases} x+y=3 \\ 0=0 \end{cases}$$

$(1,2)$; $(4,-1)$; $(8,-5)$; $(1.5, 1.5)$;
$(1.25, 1.75)$; $(2.12345678, 0.8765432)$

To describe the fact that this system has infinitely many solutions

$$(x = a,\ y = 3-a)$$

$(a, 3-a) \longrightarrow$ any solution to this system.

Eg. $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 7 \end{array}\right) \longrightarrow \begin{cases} x + y = 4 \\ 0 = 7 \end{cases}$

No solution b/c the second equation never holds.

Operations to produce row-equivalent matrices

1. Interchange 2 rows

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & 3 & 5 \end{array}\right)$$

2. Multiply a row by any nonzero constant

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow[R_1 \leftrightarrow 4R_1]{} \left(\begin{array}{cc|c} 4 & 12 & 20 \\ 2 & -1 & 3 \end{array}\right)$$

3. To add a constant multiple of a row to another row and replace the latter row by that.

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow[R_1 \leftrightarrow 4R_2 + R_1]{} \left(\begin{array}{cc|c} 9 & -1 & 17 \\ 2 & -1 & 3 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow[R_2 \leftrightarrow (-2)R_1 + R_2]{} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array}\right)$$

Q: Can you use the operations described above to turn the matrix $\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right)$ into a matrix of the form $\left(\begin{array}{cc|c} \boxed{1} & 0 & * \\ 0 & 1 & * \end{array}\right)$?

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow[R_2 \leftrightarrow -R_2]{} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ -2 & 1 & -3 \end{array}\right)$$

$$\xrightarrow[R_2 \leftrightarrow 2R_1 + R_2]{} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 7 & 7 \end{array}\right) \xrightarrow[R_2 \leftrightarrow \frac{1}{7}R_2]{}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1 \to -3R_2 + R_1]{} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

$$\begin{cases} x = 2 \\ y = 1 \end{cases}$$

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E.x

$$\left(\begin{array}{cc|c} \boxed{2} & \boxed{4} & 16 \\ \boxed{3} & \boxed{5} & 22 \end{array}\right) \xrightarrow{?} \left(\begin{array}{cc|c} \boxed{1} & 0 & * \\ \boxed{0} & 1 & * \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow \frac{1}{2} R_1} \left(\begin{array}{cc|c} 1 & 2 & 8 \\ \boxed{3} & 5 & 22 \end{array}\right) \xrightarrow{R_2 \leftrightarrow -3R_1 + R_2}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -1 & -2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow -1 R_2} \left(\begin{array}{cc|c} 1 & \boxed{2} & 8 \\ 0 & \boxed{1} & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow -2R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right)$$

$$\begin{cases} x = 4 \\ y = 2 \end{cases}$$

E.x.

$$\left(\begin{array}{cc|c} 10 & -2 & 6 \\ -5 & 1 & -3 \end{array}\right)$$

Use row operations to reduce this

$$\xrightarrow{R_1 \leftrightarrow \frac{1}{2} R_1} \left(\begin{array}{cc|c} 5 & -1 & 3 \\ -5 & 1 & -3 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_2 + R_1} \left(\begin{array}{cc|c} 5 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

$(x = a, y = 5a - 3)$

$5x - y = 3$

$y = 5x - 3$

→ infinitely many solutions

3 possible final matrix form for a linear system of 2 equations in 2 variables

I $\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right) \longrightarrow$ system has a unique solution

$x = a$; $y = b$

II $\left(\begin{array}{cc|c} 1 & m & a \\ 0 & 0 & 0 \end{array}\right) \longrightarrow$ infinitely many solutions

III $\left(\begin{array}{cc|c} 1 & m & a \\ 0 & 0 & b \end{array}\right) \longrightarrow$ No solutions.

$b \neq 0$

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E.g.

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 4 \\ 1 & 5 & 9 & 2 \\ 2 & 1 & 3 & 7 \end{array}\right)$$

$$\begin{cases} 2x + 4y + 6z = 4 \\ x + 5y + 9z = 2 \\ 2x + y + 3z = 7 \end{cases}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & \boxed{0} & \boxed{0} & * \\ \boxed{0} & \boxed{1} & \boxed{0} & * \\ \boxed{0} & \boxed{0} & \boxed{1} & * \end{array}\right)$$