6.2. Linear Programming – The Simplex Method

Monday, March 5, 2018 12:32 PM

Goal: Apply the simplex method to solve maximization problems with constraints of the form $\leq$

Recall: Last time, we used the geometric method for linear programming to solve maximization problem.

In particular, Maximize $P = 5x + 10y$ (objective function)

Subject to the constraints:

$$\boxed{\begin{array}{l} 8x + 8y \leq 160 \\ 4x + 12y \leq 180 \end{array}} \; ; \; x, y \geq 0.$$

Geometric method: (1) Graph the feasible region determined by these constraints.

(2) Find the corner points of the feasible region.

(3) Plug these corner points into the objective function

to determine the optimal solution.

Solution: $x = 7.5$; $y = 12.5$.

Maximum $P = 162.5$.

→ If the system is larger, i.e., if there are 3 or more variables, the geometric method may not work

→ **Simplex Method**

Apply the simplex method to maximize

$\boxed{P = 5x + 10y.}$ → $-5x - 10y + P = 0$

Subject to: $\boxed{8x + 8y \leq 160}$

$\boxed{4x + 12y \leq 180}$.

Step 1: Introduce Slack Variables.

Rename x to x_1 and y to x_2

The constraint has 2 inequalities → need 2 slack variables: s_1, s_2

$$8x_1 + 8x_2 + s_1 = 160$$
$$4x_1 + 12x_2 + s_2 = 180$$
$$-5x_1 - 10x_2 + P = 0$$

$x_1 \geq 0$; $x_2 \geq 0$; $s_1 \geq 0$; $s_2 \geq 0$

x_1 , x_2 : non-basic variables.

s_1, s_2, P : basic variables.

Step 2: Form the Simplex Tableau.

→ Form a 3-by-5 coefficient matrix, augmented it by the right hand side.

Pivot row Pivot Column Pivot Position

$$
\begin{array}{c|ccccc|c}
 & x_1 & x_2 & s_1 & s_2 & P & \\
\hline
s_1 & 8 & 8 & 1 & 0 & 0 & 160 \\
s_2 & 4 & \boxed{12} & 0 & 1 & 0 & 180 \\
P & -5 & -10 & 0 & 0 & 1 & 0
\end{array}
\qquad
\begin{array}{l}
\frac{160}{8} = 20 \\
\frac{180}{12} = 15
\end{array}
$$

Simplex Tableau.

<u>Step 3:</u> Find the pivot column, pivot row and pivot position of the Simplex Tableau.

* Find pivot column:

<u>Q:</u> Are there any negative #'s in the bottom row?

→ If there are none, you are done! The rightmost column is the solution.

→ Yes, there are → find the most negative #

→ -10 → column 2 is the pivot column

(Pivot column is the one containing the most negative # in the bottom row)

* Find pivot row:

To find the pivot row, divide the #'s above the -10 into the corresponding numbers in the rightmost column and find the smallest quotient.

The row that corresponds the smallest quotient is the pivot row.

→ The pivot position is the entry on the pivot row and pivot column.

Step 4: Use basic row operations to obtain the #1 the pivot position and use that to obtain 0 everywhere else in the pivot column.

exiting entering

$$
\begin{array}{c}
 \\ s_1 \\ s_2 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\
\left(\begin{array}{ccccc|c}
8 & 8 & 1 & 0 & 0 & 160 \\
4 & \boxed{12} & 0 & 1 & 0 & 180 \\
-5 & -10 & 0 & 0 & 1 & 0
\end{array}\right)
\end{array}
\xrightarrow{R_2 \leftrightarrow \frac{1}{12}R_2}
$$

$$
\left(\begin{array}{ccccc|c}
8 & 8 & 1 & 0 & 0 & 160 \\
\frac{1}{3} & \boxed{1} & 0 & \frac{1}{12} & 0 & 15 \\
-5 & -10 & 0 & 0 & 1 & 0
\end{array}\right)
\xrightarrow[R_3 \leftrightarrow 10R_2 + R_3]{R_1 \leftrightarrow -8R_2 + R_1}
$$

$$
\begin{array}{c}
 \\ s_1 \\ x_2 \\ P
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & P \end{array} \\
\left(\begin{array}{ccccc|c}
\frac{16}{3} & 0 & 1 & -\frac{2}{3} & 0 & 40 \\
\frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\
-\frac{5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150
\end{array}\right)
\end{array}
$$

the entering variable now replaces the exiting variable.

<u>Step 5:</u> If there are still negative #'s in the bottom row, repeat the process (Step 2 through 4) until there are no more negative number in the bottom row.

exiting entering pivot column pivot position

Monday, March 5, 2018 1:23 PM

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} s_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} \boxed{\frac{16}{3}} & 0 & 1 & -\frac{2}{3} & 0 & 40 \\ \frac{1}{3} & 1 & 0 & \frac{1}{12} & 0 & 15 \\ -\frac{5}{3} & 0 & 0 & \frac{5}{6} & 1 & 150 \end{array}\right) \end{array}$$

smallest quotient

$\frac{40}{16/3} = 7.5$

$\frac{15}{1/3} = 45$

I $R_1 \leftrightarrow \frac{3}{16} R_1$ $\longrightarrow$

$-\frac{1}{3}R_1 + R_2 \leftrightarrow R_2$ II
$\longrightarrow$
$\frac{5}{3}R_1 + R_3 \leftrightarrow R_3$ III

$$\begin{array}{c|c} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ P \end{array} & \left(\begin{array}{ccccc|c} 1 & 0 & \frac{3}{16} & -\frac{1}{8} & 0 & \frac{15}{2} = 7.5 \\ 0 & 1 & -\frac{1}{16} & \frac{1}{8} & 0 & \frac{25}{2} = 12.5 \\ 0 & 0 & \frac{5}{16} & \frac{5}{8} & 1 & \frac{325}{2} = 162.5 \end{array}\right) \end{array}$$

Step 6: Once there are no more negative in the bottom row, the rightmost column gives us the solution.

$x_1 = 7.5$; $x_2 = 12.5$; max $P = 162.5$

$(x = 7.5)$; $(y = 12.5)$