6.3. The Dual Problem

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Goal: Solve minimization problem with constraints of the form $\geq$.

Recap: The simplex method helps us solve maximization problem with constraints of the form $\leq$.

For e.g., $\boxed{\text{Maximize}}$ $P = 5x + 10y$

Subject to: $8x + 8y \boxed{\leq} 160$

$4x + 12y \boxed{\leq} 180$.

Today, the problem is:

$\boxed{\text{Minimize}}$ $\boxed{C = 16x_1 + 9x_2 + 21x_3.}$

Subject to: $x_1 + x_2 + 3x_3 \boxed{\geq} 12.$

$2x_1 + x_2 + x_3 \boxed{\geq} 16.$

$x_1, x_2, x_3 \geq 0$

Key idea for solving these minimization problems: to translate this back into maximization problems with $\leq$ constraints and apply the Simplex Method.

Here are the steps:

Step 1: Write down the Initial Matrix for the problem

(Coefficients for the objective function must be in the bottom row)

$$\text{Initial Matrix } A = \begin{pmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \end{pmatrix}.$$

a 3-by-4 matrix.

Step 2: Find the transpose of the initial matrix.

The transpose of a matrix A is another matrix, denoted by A^T. Columns of $A \rightarrow$ rows of A^T and Rows of $A \rightarrow$ columns of A^T

$$A^T = \begin{pmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{pmatrix}$$; a 4-by-3 matrix.

Step 3: Use the transpose of the initial matrix to rewrite the minimization problem into a maximization problem.

→ this process is called forming the dual problem

$$\begin{pmatrix} \boxed{1 \quad 2 \quad 16} \\ \boxed{1 \quad 1 \quad 9} \\ \boxed{3 \quad 1 \quad 21} \\ \boxed{12 \quad 16 \quad 1} \end{pmatrix}$$

→ $y_1 + 2y_2 \leq 16$

→ $y_1 + y_2 \leq 9$

→ $3y_1 + y_2 \leq 21$

$y_1, y_2 \geq 0.$

→ Objective function:

$P = 12y_1 + 16y_2$ → Maximize

→ The dual problem is :

Maximize $P = 12y_1 + 16y_2$.

Subject to the constraints:

$$y_1 + 2y_2 \le 16$$

$$y_1 + y_2 \le 9$$

$$3y_1 + y_2 \le 21$$

We can apply the simplex method to solve this.

Step 4: Apply the Simplex method from last time to solve this dual problem.

* Introduce slack variables
* Form the Simplex Tableau.
* Identify pivot positions.
* Apply row operations to reduce simplex tableau.
* Keep track of exiting and entering variable.

$$
\begin{aligned}
y_1 + 2y_2 + x_1 &= 16\\
y_1 + y_2 + x_2 &= 9\\
3y_1 + y_2 + x_3 &= 21\\
-12y_1 - 16y_2 + P &= 0
\end{aligned}
$$

enter

pivot position

pivot row

exit

$$
\begin{array}{c|cccccc|c}
 & y_1 & y_2 & x_1 & x_2 & x_3 & P & \\
\hline
x_1 & 1 & 2 & 1 & 0 & 0 & 0 & 16 \\
x_2 & 1 & \boxed{1} & 0 & 1 & 0 & 0 & 9 \\
x_3 & 3 & \boxed{1} & 0 & 0 & 1 & 0 & 21 \\
P & -12 & -16 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

$\frac{16}{2} = 8$

$\frac{9}{1} = 9$

$\frac{21}{1} = 21$

pivot column

$$\xrightarrow{\frac{1}{2}R_1} \quad \xrightarrow[-1R_1 + R_3 \to R_3]{-1R_1 + R_2 \to R_2} \quad \xrightarrow{16R_1 + R_4 \to R_4}$$

$$\begin{array}{c|c} & \begin{array}{cccccc|c} y_1 & y_2 & x_1 & x_2 & x_3 & P & \end{array} \\ \begin{array}{c} y_2 \\ x_2 \\ x_3 \\ P \end{array} & \left(\begin{array}{cccccc|c} \boxed{\frac{1}{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 8 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 1 \\ \boxed{\frac{5}{2}} & 0 & -\frac{1}{2} & 0 & 1 & 0 & 13 \\ \boxed{-4} & 0 & 8 & 0 & 0 & 1 & 128 \end{array}\right) \end{array}$$

pivot

$8/1/2 = 16$

$1/1/2 = 2$ pivot row

$13/5/2 = 5.2$

$\xrightarrow{2R_2}$ $\xrightarrow{-\frac{1}{2}R_2 + R_1 \leftrightarrow R_1}$ $\xrightarrow{-\frac{5}{2}R_2 + R_3 \leftrightarrow R_3}$ $\xrightarrow{4R_2 + R_4 \leftrightarrow R_4}$

→ pivot position → turn it to 1 → row operations

$$\begin{array}{c|c} & \begin{array}{cccccc|c} y_1 & y_2 & x_1 & x_2 & x_3 & P & \end{array} \\ \begin{array}{c} y_2 \\ y_1 \\ x_3 \\ P \end{array} & \left(\begin{array}{cccccc|c} 0 & 1 & 1 & -1 & 0 & 0 & 7 \\ 1 & 0 & -1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & -5 & 1 & 0 & 8 \\ 0 & 0 & 4 & 8 & 0 & 1 & 136 \end{array}\right) \end{array}$$

Solution to the dual problem is

$y_1 = 2$; $y_2 = 7$; max $P = 136$.

Step 5: The Solution to the original minimization problem can be read off from the bottom row of the tableau.

$$\boxed{\begin{aligned}&\text{So, } x_1 = 4\ ;\ x_2 = 8\ ;\ x_3 = 0\\ &\text{Min } C = 136\end{aligned}}$$