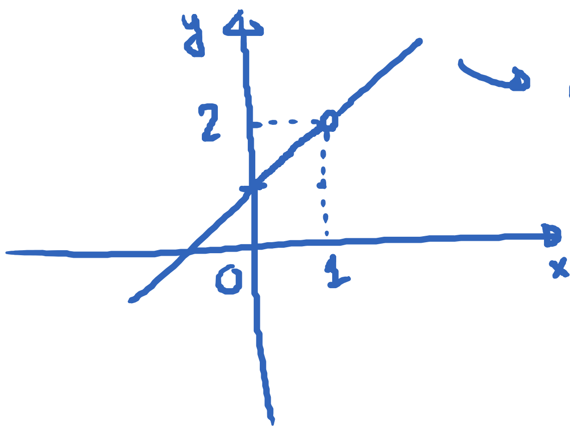
But for many situations, we will not get a finite #, it does not mean that the limit DNE.

E.g. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ Plug $x = 1$ to the function; we get : $\dfrac{0}{0}$

$$\frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x + 1$$

provided that $x \neq 1$

$$\frac{x^2-1}{x-1} = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



$\rightarrow$ graph of $\dfrac{x^2-1}{x-1}$

We see that

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 = \lim_{x \to 1} (x + 1)$$

This suggests the following technique to find limits of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ when we get $\dfrac{0}{0}$ if we plug $x = a$ to $\dfrac{f(x)}{g(x)}$.

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1. Factor the top and bottom functions completely.
2. Cancel the common factors.
3. Plug $x = a$ into the simplified function.

Ex. (1) $\lim_{x \to -3} \dfrac{x^2 + 4x + 3}{x^2 - 9} = \dfrac{1}{3}$

(2) $\lim_{h \to 0} \dfrac{(1+h)^2 - 1}{h} = 2.$

Some Variations of the $\frac{0}{0}$ limit type.

$\frac{0}{0}$ limits that involve radicals.

E.g. $\lim_{x \to 5} \dfrac{\sqrt{x-1} - 2}{x - 5} \quad \left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 5} \frac{\overset{A}{\sqrt{x-1}} - \overset{B}{2}}{x-5} \cdot \frac{\overset{A}{\sqrt{x-1}} + \overset{B}{2}}{\sqrt{x-1} + 2}$$

* limits of the form $\frac{K}{0}$ where $K \neq 0$.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \; ; \; \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Ex. $$\lim_{x \to 1} \frac{x+2}{(x-1)^2} = \infty$$

$$\lim_{x \to 1} \frac{x-2}{(x-1)^2} = -\infty \; ; \; \lim_{x \to 1} \frac{2-x}{(x-1)^2} = \infty$$

$$\lim_{x \to 1^+} \frac{x-2}{(x-1)^3} = -\infty$$

$$\underbrace{\lim_{x \to 1} \frac{x-2}{(x-1)^3}}_{\text{DNE}}$$

$$\lim_{x \to 1^-} \frac{x-2}{(x-1)^3} = \infty$$

* One-sided limits

$$f(x) = \begin{cases} -x-2 & \text{if } \boxed{x < -1} \\ 2 & \text{if } x = -1 \\ x^3 & \text{if } x > -1 \end{cases}$$

(piecewise - function)

$$= \lim_{x\to 5} \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x-1}+2)}$$

$$= \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2} = \boxed{\frac{1}{4}}$$

Ex. $\displaystyle \lim_{t\to 9} \frac{t-9}{\sqrt{t}-3} = 6$

* limits that involve complex fractions

E.g. $$\lim_{h\to 0} \frac{\frac{5}{5}\cdot\frac{1}{(5+h)} - \frac{1}{5}\cdot\frac{(5+h)}{5+h}}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{\frac{5-(5+h)}{5\cdot(5+h)}}{h} = \lim_{h\to 0} \frac{\boxed{\frac{-h}{5\cdot(5+h)}}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{5\cdot(5+h)} \cdot \frac{1}{\cancel{h}} = \lim_{h\to 0} \frac{-1}{5\cdot(5+h)} = -\frac{1}{25}$$

$$\lim_{x \to -1} (x^3) = -1$$

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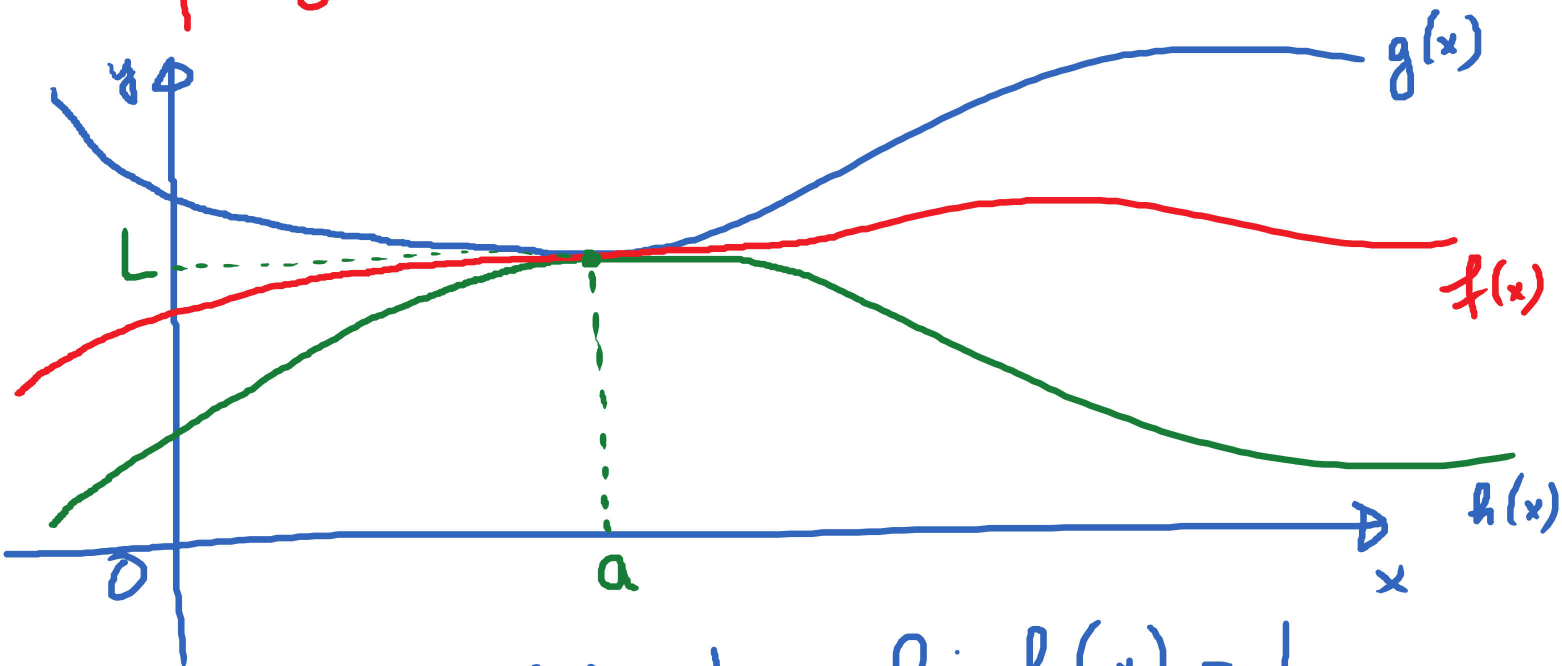
$$\underbrace{\lim_{x \to -1^-} f(x)}_{-1} \; ; \; \underbrace{\lim_{x \to -1^+} f(x)}_{-1} \; ; \; \underbrace{\lim_{x \to -1} f(x)}_{-1}$$

$$f(-1) = 2$$

$\rightarrow -1 \leftarrow$

$$= \lim_{x \to -1} (-x-2) = 1 - 2 = -1$$

* Squeeze Theorem



Given: $\lim_{x \to a} g(x) = L \; ; \; \lim_{x \to a} h(x) = L$

$h(x) \leq f(x) \leq g(x)$ (near a)

$x \to a$

$$h(x) \leq f(x) \leq g(x) \quad (\text{near } a)$$

Then: $\boxed{\lim_{x \to a} f(x) = L}$

$$h(x) \le f(x) \le g(x)$$

as $x \to a$

$$L$$

* Using Squeeze Theorem to find

$$\lim_{x \to 0} \frac{\sin x}{x} \quad \left(\frac{0}{0}\right)$$

It turns out that: $\boxed{\frac{\sin x}{x}} < \boxed{1}$

$f(x)$ $g(x)$

$h(x)$

$\frac{\sin x}{x} > \boxed{\cos x}$ (near 0)

$$\cos x < \frac{\sin x}{x} < 1$$

when $x \to 0$ $\quad 1$

By Squeeze Theorem: $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

E.g. Apply the limit $\lim_{x \to 0} \frac{\sin x}{x}$ to find other related limits

$$\lim_{x \to 0} \frac{\sin(2018x)}{x} \qquad \frac{0}{0}$$

let $u = 2018x$

$$\lim_{u \to 0} \frac{\sin(u)}{\frac{u}{2018}} = \lim_{u \to 0} \sin(u) \cdot \frac{2018}{u}$$

$$= \lim_{u \to 0} 2018 \cdot \frac{\sin(u)}{u}$$

$$= 2018 \cdot \boxed{\lim_{u \to 0} \frac{\sin(u)}{u}} = 1$$

$$= 2018 \cdot 1 = \boxed{2018}$$

Ex. $\lim_{x \to 0} \frac{\sin(7x)}{3x}$