2.4. Continuity

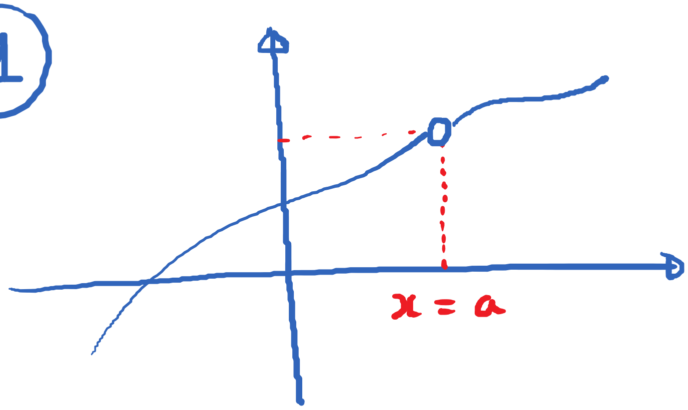
Monday, January 29, 2018 8:00 AM

Goal: Understand the concept of continuity rigorously using limits.

* What does it mean for a function to be continuous at a point $x = a$?

→ When does a function f fail to be continuous at a point $x = a$.

1

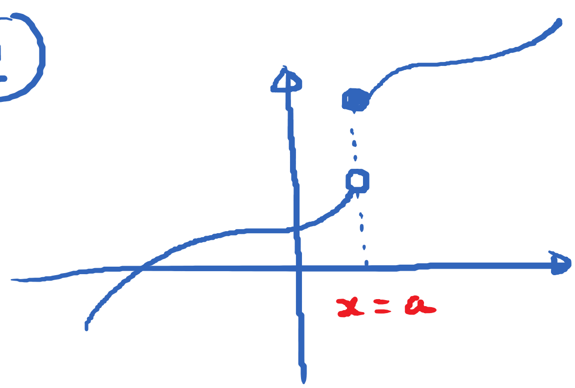


If $f(a)$ is undefined, then is not continuous at $x = a$.

→ 1^{st} requirement for f to be continuous at $x = a$:

$\boxed{f(a) \text{ must be defined}}$

2



If $\lim_{x \to a} f(x)$ DNE, then f is not continuous at a.

→ 2^{nd} requirement for continuity

$\boxed{\lim_{x \to a} f(x) \text{ must exist}}$

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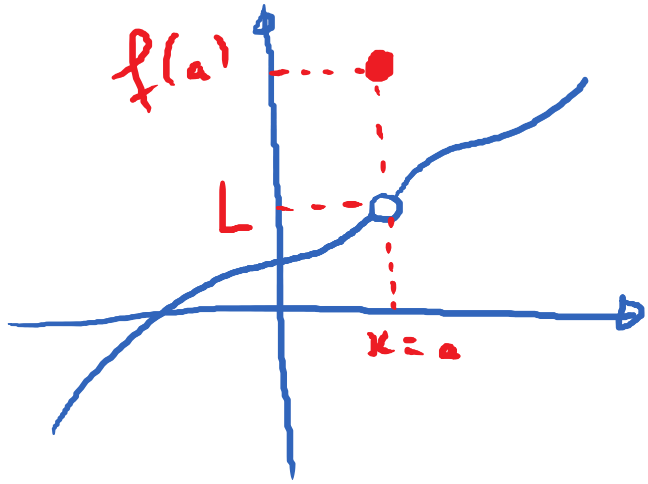


3^{rd} requirement for continuity

$$\lim_{x \to a} f(x) = f(a)$$

Definition of continuity at a point:

A function f is continuous at a point $x = a$ if and only if the following conditions are satisfied

1. $f(a)$ must be defined
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$

E.g. let $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$

Use the above definition to verify that f is continuous at the point $x = 0$.

(1) Is $f(0)$ defined? ✓ $f(0) = -1$

(2) ~~Does~~ $\lim_{x \to 0} f(x)$ exist? ✓

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0} (x^2 - e^x) = -1. \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0} (x - 1) = -1 \end{aligned}\right\}$$

$\rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \quad (= -1)$

Therefore, $\lim_{x \to 0} f(x)$ exists.

(3) Is $f(0) = \lim_{x \to 0} f(x)$? ✓

Since (1), (2) and (3) are satisfied, f is continuous at $x = 0$.

Eg. $f(x) = \frac{1}{x}$.

f is not continuous at $x = 0$.

Why? $f(0)$ is undefined. -150

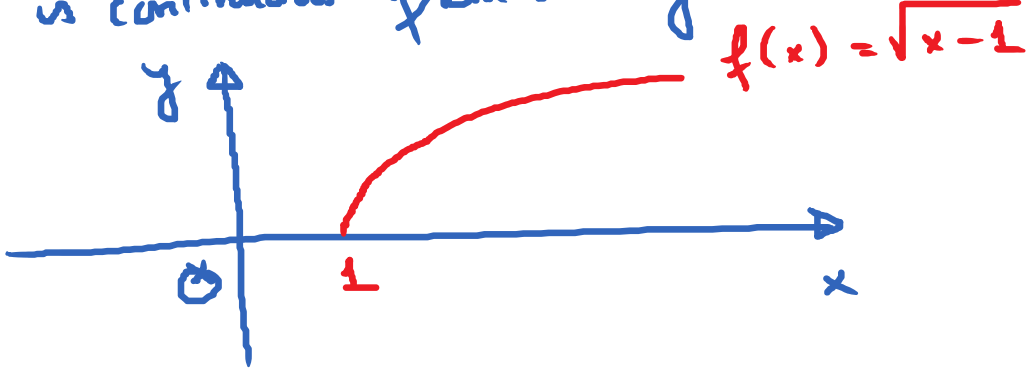
Eg. on the interval: $(-\infty, 1)$

$f(x) = \sqrt{x-1}$

Q: Is f continuous at $x = 1$?

1. $f(1)$ is defined. $f(1) = 0$
2. $\lim_{x \to 1^+} f(x) = 0$
3. $f(1) = \lim_{x \to 1^+} f(x)$

f is continuous from the right at $x = 1$



E.g.

$$w(x) = \begin{cases} x-1 & \text{if } x>0 \\ 7 & \text{if } x=0 \\ x+3 & \text{if } x<0 \end{cases}$$

$\lim_{x\to 0^-} w(x) = 3$; $\lim_{x\to 0^+} w(x) = -1$

$\lim_{x\to 0} w(x)$ DNE b/c left limit $\neq$ right limit

So f is not continuous b/c 2 is not satisfied.

E.g.

$$f(x) = \begin{cases} -x-3 & \text{if } x>0 \\ 10000 & \text{if } x=0 \\ x-3 & \text{if } x<0 \end{cases}$$

$f(0) = 10000$

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0}(x-3) = -3 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0}(-x-3) = -3 \end{aligned}\right\}$$

So, $\lim_{x\to 0} f(x)$ exists and equals -3

$\lim_{x\to 0} f(x) \neq f(0)$

3 is not satisfied

f is not cont. at $x=0$

Ex. (1) Find the number g such that f is continuous at $x = 0$ where f is the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ g & \text{if } x = 0 \end{cases}$$

(2) Find the number i such that f is continuous at $x = -2$.

$$f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 2} & \text{if } x \neq -2 \\ i & \text{if } x = -2 . \end{cases}$$

(1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

$f(0) = g$.

So, for f to be continuous at $x = 0$, we must have:

$$f(0) = \lim_{x \to 0} f(x)$$

$$\boxed{g = 1}$$

2

$$\lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+1)\cancel{(x+2)}}{\cancel{x+2}}$$

$$= \lim_{x\to -2} (x+1) = -1$$

$f(-2) = i$.

Therefore, for f to be continuous at $x=-2$,

$$f(-2) = \lim_{x\to -2} f(x)$$

$$\boxed{i = -1}$$

Definition: We say that f is continuous on an interval I. ($I=(a,b)$, $I=[a,b]$, $I=(-\infty, a)$ ...)

if f is continuous at every point within that interval.

Theorem: Polynomial Functions, rational functions, exponential functions, radical functions are continuous everywhere in their domain.