3.6 The Chain Rule

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$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left[(\text{Stuff})^{\boxed{3}}\right] = 3(\text{Stuff})^2 \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left[\sqrt{\text{Stuff}}\right] = \frac{1}{2\sqrt{\text{Stuff}}} \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left[\frac{1}{\text{Stuff}}\right] = -\frac{1}{(\text{Stuff})^2} \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left[\cos(\text{Stuff})\right] = -\sin(\text{Stuff}) \cdot \frac{d}{dx}(\text{Stuff})$$

$$\frac{d}{dx}\left[\Big(\underbrace{\boxed{\cos x + x^2 + 7}}_{\text{Stuff}}\Big)^3\right] = 3\left(\cos x + x^2 + 7\right)^2 \cdot \left(-\sin x + 2x\right)$$

$$\frac{d}{dx}\left[\sqrt{x^2 + 4x - 7}\right] = \frac{2x + 4}{2\sqrt{x^2 + 4x - 7}}$$

$$\frac{d}{dx}\left[\sec\left(x^3+3\cos x\right)\right] =$$

$$\sec\left(x^3+3\cos x\right)\tan\left(x^3+3\cos x\right)\cdot\left[3x^2-3\sin x\right].$$

Chain Rule (2-component chain)

If y is a function of u and u is a function of x, then: $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$\begin{tikzcd} y \arrow[r, dash, bend left, "\frac{dy}{du}"] \arrow[rr, dash, bend right] & u \arrow[r, dash, bend left, "\frac{du}{dx}"] & x \end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

E.g. $\frac{d}{dx}\left[\sqrt[3]{\sin x + \cos x}\right]$

$y = \sqrt[3]{u} = (u)^{\frac{1}{3}}$

$\boxed{\frac{dy}{du}} = \frac{1}{3}(u)^{-2/3}$

$u = \sin x + \cos x$

$\boxed{\frac{du}{dx}} = \cos x - \sin x$

$y = \sqrt[3]{\sin x + \cos x} = \sqrt[3]{u}$

$u = \sin x + \cos x$

$y \text{ — } u \text{ — } x$

$\frac{dy}{dx} = \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dx}} = \frac{1}{3}(u)^{-2/3} \cdot (\cos x - \sin x)$

$\frac{dy}{dx} = \frac{1}{3}(\sin x + \cos x)^{-2/3} \cdot (\cos x - \sin x)$

E.g. $\frac{d}{dx}\left[(x^2 + 5x + 2)^{3/2}\right]$

$y \text{ — } u \text{ — } x$

$y = u^{3/2}$

$\frac{dy}{du} = \frac{3}{2}u^{1/2}$

$u = x^2 + 5x + 2$

$\frac{du}{dx} = 2x + 5$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{3}{2}u^{1/2} \cdot (2x+5)$$

$$= \frac{3}{2}\left(x^2+5x+2\right)^{1/2} \cdot (2x+5)$$

$$= \frac{3}{2}\sqrt{x^2+5x+2} \cdot (2x+5)$$

Chain Rule with more than 2 components

$$y \text{ —— } u \text{ —— } v \text{ —— } w \text{ —— } x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

E.g. $y = \sin\left(\boxed{\cos\left(\boxed{\sqrt{x}}\right)}\right)$. $\dfrac{dy}{dx} \stackrel{?}{=}$

$y = \sin(u)$

$u = \cos(v)$

$v = \sqrt{x}$

$$y \text{ —— } u \text{ —— } v \text{ —— } x$$

dy/dx

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos(u) \cdot \left(-\sin(v)\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= \cos(\cos(\sqrt{x})) \left(-\sin(\sqrt{x})\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2} \cos(\cos(\sqrt{x})) \cdot \sin(\sqrt{x}) \cdot \frac{1}{\sqrt{x}}$$

Ex. 1 $y = \boxed{\sin\left(\boxed{\sin\left(\boxed{\sin x}\right)}\right)}$. $\frac{dy}{dx} = ?$

2 $y = \cos\left(\sqrt{\sin(\tan(\pi x))}\right)$. $\frac{dy}{dx} = ?$

3 $y = \left[x + \left(x + \sin^2 x\right)^3\right]^4$. $\frac{dy}{dx} = ?$

$v = \sin x \longrightarrow \frac{dv}{dx} = \cos(x)$

$u = \sin(v) \longrightarrow \frac{du}{dv} = \cos(v)$

$y = \sin(u) \rightarrow \frac{dy}{du} = \cos(u)$

$$\frac{dy}{dx} = \cos(u) \cdot \cos(v) \cdot \cos x$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$$

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(2) $\cos\left(\boxed{\sqrt{\boxed{\sin\left(\boxed{\tan\left(\boxed{\pi x}\right)}\right)}}}\right)$

$p = \pi x$

$c = \tan(p)$

$m = \sin(c)$

$r = \sqrt{m}$

$y = \cos(r)$

$$y - r - m - c - p - x$$

$$\frac{dy}{dx} =$$

$$\frac{dy}{dr} = -\sin(r) \; ; \quad \frac{dr}{dm} = \frac{1}{2\sqrt{m}} \; ; \quad \frac{dm}{dc} = \cos c \; ;$$

$$\frac{dc}{dp} = \sec^2(p) \; ; \quad \frac{dp}{dx} = \pi$$

$$\frac{dy}{dx} = -\sin(r) \cdot \frac{1}{2\sqrt{m}} \cdot \cos x \cdot \sec^2(p) \cdot \pi$$

$$\boxed{} \frac{1}{2\sqrt{\sin(\tan(\pi x))}} \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi$$

$$-\sin\left(\sqrt{\sin(\tan(\pi x))}\right)$$

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3) $y = \left[\boxed{x + \left(\boxed{x + \sin^2 x}\right)^3}\right]^4$

u

$y = u^4$

$y — u — x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \cdot \frac{du}{dx} = 4\left[x + (x + \sin^2 x)^3\right]^3 \cdot \frac{du}{dx}$$

Find $\frac{du}{dx}$. $\quad u = x + v^3$

$$\frac{du}{dx} = 1 + \frac{d}{dx}\left[v^3\right]$$

$$\frac{d}{dx}\left[v^3\right] = 3v^2 \cdot \frac{dv}{dx}$$

$$= 3\left(x + \sin^2 x\right)^2 \cdot \frac{dv}{dx} \cdot \square$$

Find $\frac{dv}{dx}$. $\quad v = x + \sin^2 x$

$$\frac{dv}{dx} = 1 + \frac{d}{dx}\left[\sin^2 x\right]$$

$$= \boxed{1 + 2\sin x \cdot \cos x}$$

$$\frac{d}{dx}\left[v^3\right] = 3\left(x + \sin^2 x\right)^2 \cdot \left(1 + 2\sin x \cos x\right)$$

$$\frac{du}{dx} = \underbrace{1 + 3\left(x + \sin^2 x\right)^3 \cdot \left(1 + 2\sin x \cos x\right)}$$

$$\frac{dy}{dx} = 4\left[x + \left(x + \sin^2 x\right)^3\right]^3 \cdot \boxed{}$$

In Newton's Notation. (Chain Rule)

$$y = f(\underbrace{\boxed{g(x)}}_{u})$$

$$y = f(u)$$

$$u = g(x)$$

$$\boxed{y' = f'(g(x)) \cdot g'(x)}$$