

# 3.6 The Chain Rule

Monday, February 12, 2018

8:23 AM

$$\frac{d}{dx} [x^3] = 3x^2$$

$$\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2}$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\frac{d}{dx} \left[ (\text{Stuff})^3 \right] = 3(\text{Stuff})^2 \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} \left[ \sqrt{\text{Stuff}} \right] = \frac{1}{2\sqrt{\text{Stuff}}} \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} \left[ \frac{1}{\text{Stuff}} \right] = -\frac{1}{(\text{Stuff})^2} \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} [\cos(\text{Stuff})] = -\sin(\text{Stuff}) \cdot \frac{d}{dx} (\text{Stuff})$$

$$\frac{d}{dx} \left[ (\underbrace{\cos x + x^2 + 7}_{\text{Stuff}})^3 \right] = 3(\cos x + x^2 + 7)^2 \cdot (-\sin x + 2x)$$

$$\frac{d}{dx} \left[ \sqrt{x^2 + 4x - 7} \right] = \frac{2x + 4}{2\sqrt{x^2 + 4x - 7}}$$

$$\frac{d}{dx} \left[ \sec(x^3 + 3\cos x) \right] = \sec(x^3 + 3\cos x) \tan(x^3 + 3\cos x) \cdot [3x^2 - 3\sin x]$$


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Chain Rule (2-component chain)

If  $y$  is a function of  $u$  and  $u$  is a function of  $x$ , then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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E.g.  $\frac{d}{dx} \left[ \sqrt[3]{\sin x + \cos x} \right]$

$$y = \sqrt[3]{\sin x + \cos x} = \sqrt[3]{u}$$

$$u = \sin x + \cos x$$

Diagram showing the relationship between  $y$ ,  $u$ , and  $x$ . A horizontal line has  $y$  at the left end,  $u$  in the middle, and  $x$  at the right end. A red arrow points from  $u$  to the  $\frac{dy}{du}$  term in the chain rule below. A green arrow points from  $x$  to the  $\frac{du}{dx}$  term in the chain rule below.

$$y = \sqrt[3]{u} = (u)^{\frac{1}{3}}$$

$$\frac{dy}{du} = \frac{1}{3} (u)^{-2/3}$$

$$u = \sin x + \cos x$$

$$\frac{du}{dx} = \cos x - \sin x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{3} (u)^{-2/3} \cdot (\cos x - \sin x)$$

$$\frac{dy}{dx} = \frac{1}{3} (\sin x + \cos x)^{-2/3} \cdot (\cos x - \sin x)$$

E.g.  $\frac{d}{dx} \left[ (x^2 + 5x + 2)^{3/2} \right]$

$$y = u^{3/2}$$

$$u = x^2 + 5x + 2$$

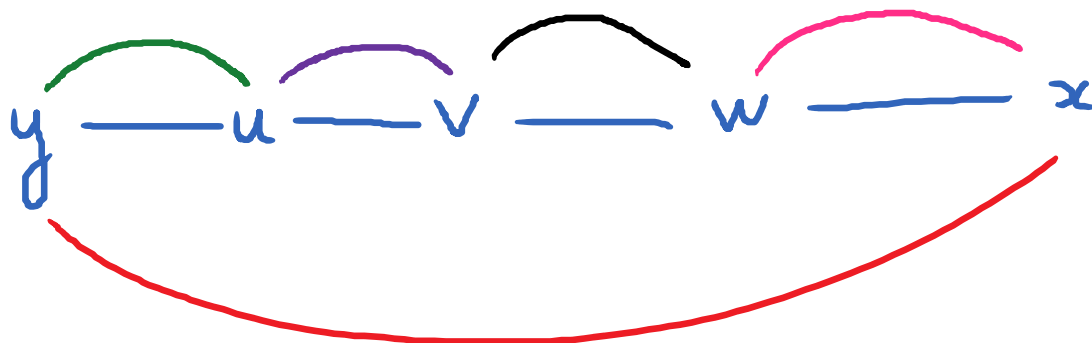
$$\frac{dy}{du} = \frac{3}{2} u^{1/2}$$

$$\frac{du}{dx} = 2x + 5$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{3}{2} u^{1/2} \cdot (2x+5) \\
 &= \frac{3}{2} (x^2 + 5x + 2)^{1/2} \cdot (2x+5) \\
 &= \frac{3}{2} \sqrt{x^2 + 5x + 2} \cdot (2x+5)
 \end{aligned}$$


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Chain Rule with more than 2 components



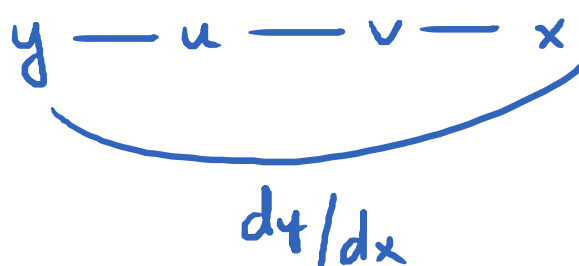
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

E.g.  $y = \sin(\cos(\sqrt{x}))$  .  $\frac{dy}{dx} = ?$

$$y = \sin(u)$$

$$u = \cos(v)$$

$$v = \sqrt{x}$$



$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos(u) \cdot (-\sin(v)) \cdot \frac{1}{2\sqrt{x}}$$

$$= \cos(\cos(\sqrt{x})) (-\sin(\sqrt{x})) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2} \cos(\cos(\sqrt{x})) \cdot \sin(\sqrt{x}) \cdot \frac{1}{\sqrt{x}}$$

Ex. ①  $y = \sin(\sin(\sin x)) \cdot \frac{dy}{dx} = ?$

②  $y = \cos(\sqrt{\sin(\tan(\pi x))}) \cdot \frac{dy}{dx} = ?$

③  $y = [x + (x + \sin^2 x)^3]^4 \cdot \frac{dy}{dx} = ?$

$v = \sin x \rightarrow \frac{dv}{dx} = \cos(x)$

$u = \sin(v) \rightarrow \frac{du}{dv} = \cos(v)$

$y = \sin(u) \rightarrow \frac{dy}{du} = \cos(u)$

$$\begin{aligned}\frac{dy}{dx} &= \cos(u) \cdot \cos(v) \cdot \cos x \\ &= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x\end{aligned}$$


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$$(2) \cos\left(\sqrt{\sin(\tan(\pi x))}\right)$$

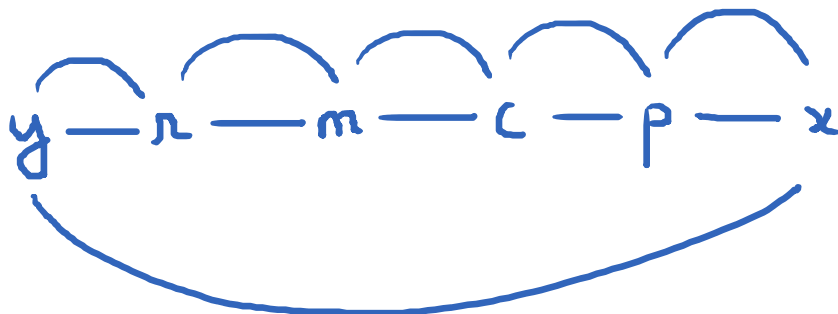
$$p = \pi x$$

$$c = \tan(p)$$

$$m = \sin(c)$$

$$n = \sqrt{m}$$

$$y = \cos(n)$$



$$\frac{dy}{dx} =$$

$$\frac{dy}{dn} = -\sin(n); \quad \frac{dn}{dm} = \frac{1}{2\sqrt{m}}; \quad \frac{dm}{dc} = \cos c;$$

$$\frac{dc}{dp} = \sec^2(p); \quad \frac{dp}{dx} = \pi$$

$$\frac{dy}{dx} = -\sin(\pi) \cdot \frac{1}{2\sqrt{m}} \cdot \cos(\pi) \cdot \sec^2(\pi) \cdot \pi$$

$$\frac{1}{2\sqrt{\sin(\tan(\pi x))}} \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi$$

$$-\sin(\sqrt{\sin(\tan(\pi x))})$$

$$(3) \quad y = \left[ x + \left( x + \sin^2 x \right)^3 \right]^4$$

$$y = u^4$$

$$y \text{ --- } u \text{ --- } x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 4u^3 \cdot \frac{du}{dx} = 4 \left[ x + (x + \sin^2 x)^3 \right]^3 \cdot \frac{du}{dx}$$

Find  $\frac{du}{dx}$ .  $u = x + v^3$

$$\frac{du}{dx} = 1 + \frac{d}{dx} [v^3]$$



$$\begin{aligned}\frac{d}{dx} [v^3] &= 3v^2 \cdot \frac{dv}{dx} \\ &= 3(x + \sin^2 x)^2 \cdot \frac{dv}{dx}\end{aligned}$$

Find  $\frac{dv}{dx}$ .  $v = x + \sin^2 x$

$$\frac{dv}{dx} = 1 + \frac{d}{dx} [\sin^2 x]$$

$$= \boxed{1 + 2 \sin x \cdot \cos x}$$

$$\frac{d}{dx} [v^3] = 3(x + \sin^2 x)^2 \cdot (1 + 2 \sin x \cos x)$$

$$\frac{du}{dx} = \frac{1}{\sqrt{\dots}} + 3(x + \sin^2 x)^3 \cdot (1 + 2 \sin x \cos x)$$

$$\frac{dy}{dx} = 4 \left[ x + (x + \sin^2 x)^3 \right]^3 \cdot \boxed{\dots}$$

In Newton's Notation. (Chain Rule)

$$y = f(\underbrace{g(x)}_u)$$

$$y = f(u)$$

$$u = g(x)$$

$$y' = f'(g(x)) \cdot g'(x)$$