Derivatives of Inverse Trig Functions

Wednesday, February 14, 2018 8:18 AM

Recall:

$\arcsin\left(\frac{1}{2}\right) \overset{?}{=} \frac{\pi}{6}$ (Range of $y = \arcsin(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)

$\arcsin(1) = \frac{\pi}{2}$

$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ (Range of $y = \arccos(x)$ is $[0, \pi]$)

$\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$. (Range of $y = \arctan(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$)

To sum up,

$\arcsin(x)$ gives us an angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin(y) = x$

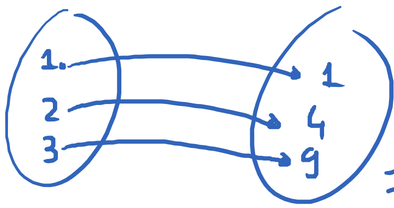
$\arccos(x)$ gives us an angle y in $[0, \pi]$ such that $\cos(y) = x$

$\arctan(x)$ gives us an angle y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(y) = x$

In general, if $y = f(x)$ is a function, then the inverse function of f, denoted by, f^{-1} (not equal to $\frac{1}{f}$) is the function

$y = x^2$

$f(x) = x^2$

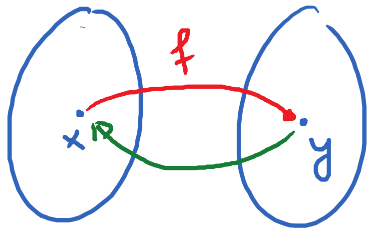


$y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}$

that undoes what f does.

$$\boxed{\begin{aligned} f^{-1}(f(x)) &= x \\ f(f^{-1}(x)) &= x \end{aligned}}$$



Inverse Function Theorem:

Let f be a function that is invertible and differentiable on an interval containing x

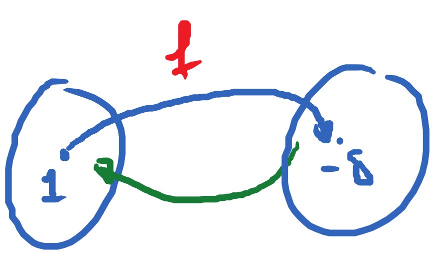
Then:

$$\boxed{\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}}$$

provided that $f'(f^{-1}(x)) \neq 0$

E.g. $\boxed{f(x) = x^5 + 3x^3 - 4x - 8}$.

$f(1) = -8$; $\boxed{f^{-1}(-8) = 1}$.



No "nice" formula for $y = f^{-1}(x)$.

Q: Find the equation of the tangent line to the graph of $y = f^{-1}(x)$ at $(-8, 1)$

Slope of tangent line at $(-8, 1)$

$$= (f^{-1})'(-8)$$

By the I.F.T. :

$$(f^{-1})'(-8) = \frac{1}{f'(f^{-1}(-8))}$$

$$= \frac{1}{f'(1)}$$

$f'(x) = 5x^4 + 9x^2 - 4 \rightarrow f'(1) = 10$.

Hence, $(f^{-1})'(-8) = \boxed{\frac{1}{10}}$ ← Slope.

Point-Slope Equation:

$$y - 1 = \frac{1}{10}(x + 8)$$

$$\boxed{y = \frac{1}{10}x + \frac{9}{5}}$$

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Why is the I.F.T. true ?

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Key: $f(f^{-1}(x)) = x$

Take the derivative w.r.t x of both sides

$$\boxed{f'(f^{-1}(x))} \cdot \boxed{(f^{-1})'(x)} = 1$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \checkmark$$

E.g. $f(x) = x^3 + 2x + 3.$

Find $(f^{-1})'(0) \stackrel{?}{=}$

$\boxed{\begin{array}{l} f(0) = 3 \\ f^{-1}(3) = 0 \end{array}}$

$f(-1) = 0$

$f^{-1}(0) = -1.$

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(-1)} = \boxed{\frac{1}{5}}$$

$$f'(x) = 3x^2 + 2$$

$$f'(-1) = 5$$

Derivatives of Inverse Trig Functions.

In the equation for the IFT:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

If rename $f^{-1}(x)$ as $g(x)$, the equation becomes:

$$g'(x) = \frac{1}{f'(g(x))}$$

(f and g are inverse functions of each other)