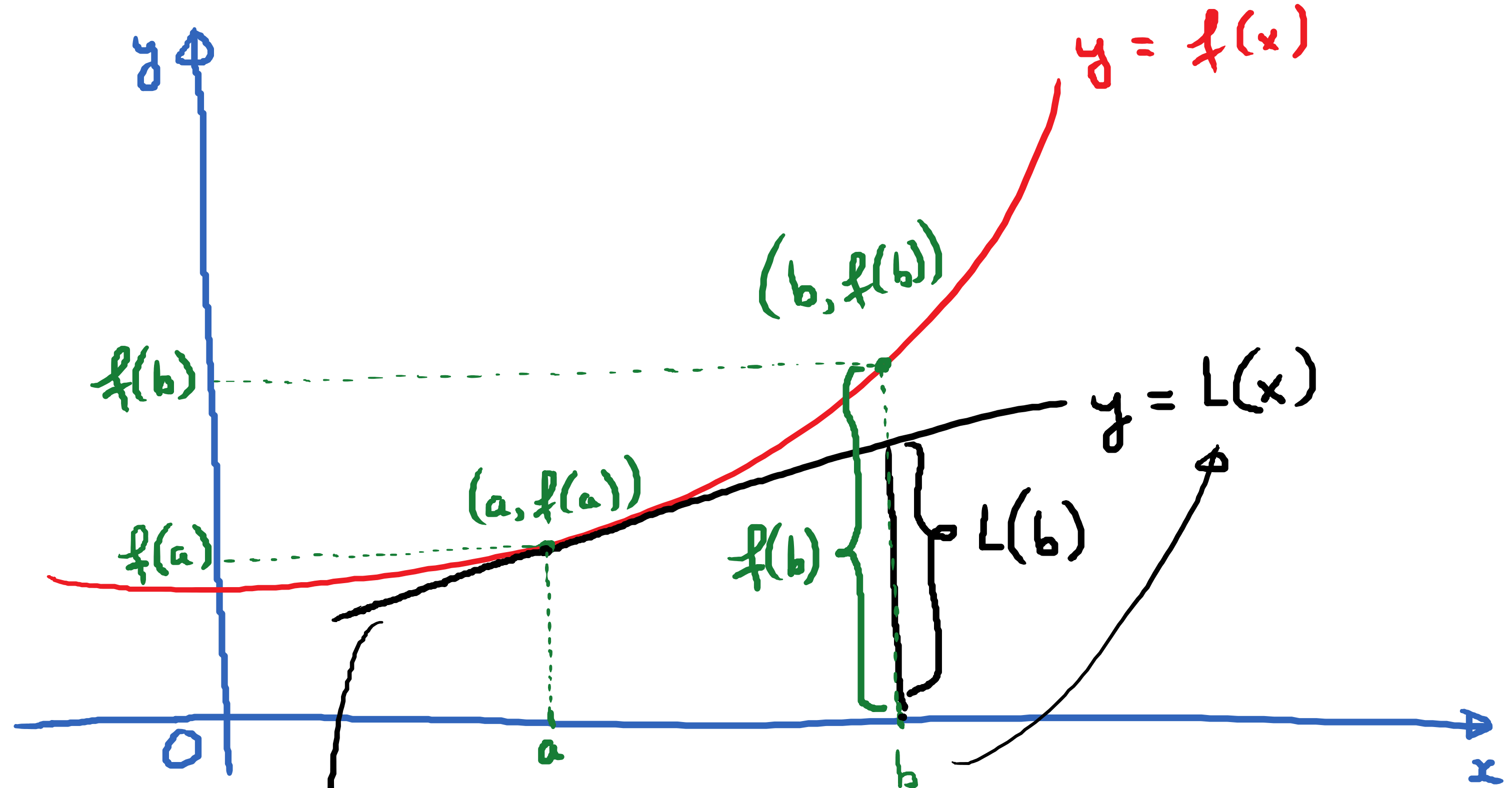
# 4.2. Linear Approximations and Differentials

Wednesday, February 28, 2018 8:35 AM

## ① Linear Approximation




tangent line to $y = f(x)$ at $(a, f(a))$

$$\boxed{L(b) \approx f(b)}$$

$L(b)$ is a linear approximation to $f(b)$.

→ Find equation for $L(x)$?

The linear approximation of the function $y = f(x)$ at the point $(a, f(a))$ is the function $y = L(x)$.

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The equation $y = L(x)$ is the equation of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

→ Formula for the linear approximation $y = L(x)$:

Slope of tangent line $= f'(a)$

Point of tangency $= (a, f(a))$

Point-Slope equation of tangent line:

$$y - f(a) = f'(a)(x - a).$$

$$\longrightarrow \quad y = f'(a)(x-a) + f(a)$$

$$\longrightarrow \quad \boxed{L(x) = f(a) + f'(a)(x-a)}$$

formula for the linear approximation to $y = f(x)$ at $(a, f(a))$.

$L(x)$

E.g.* let $f(x) = \sqrt{x}$

(a) Find the linear approximation to the graph of $f$ at $a = 9$.

(b) Use the linear approximation $L(x)$ to estimate $\sqrt{9.1}$.

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(2)* Use linear approximation to approximate $\sqrt[3]{1001}$

(1) (a) $L(x) = f(9) + f'(9)(x-9)$

$f(9) = 3$ ; $f'(x) = \dfrac{1}{2\sqrt{x}}$ ; $f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$.

$L(x) = 3 + \frac{1}{6}(x-9)$

(b) Approximate $\sqrt{9.1}$

$$\sqrt{9.1} \approx L(9.1) = 3 + \frac{1}{6}(9.1-9)$$

$$= 3 + \frac{0.1}{6} \approx 3.0167$$

$x^{\frac{1}{3}}$

② $f(x) = \sqrt[3]{x}$ ; $a = 1000$.

$$L(x) = f(1000) + f'(1000)(x - 1000)$$

$$f(1000) = 10; \quad f'(x) = \frac{1}{3} \cdot x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(1000) = \frac{1}{3\left(\sqrt[3]{1000}\right)^2} = \frac{1}{300}.$$

$$L(x) = 10 + \frac{1}{300} \cdot (x - 1000)$$

$$\sqrt[3]{1001} \approx 10 + \frac{1}{300} \cdot (1001 - 1000)$$

$$10 + \frac{1}{300} = \frac{3001}{300} \simeq \ldots$$

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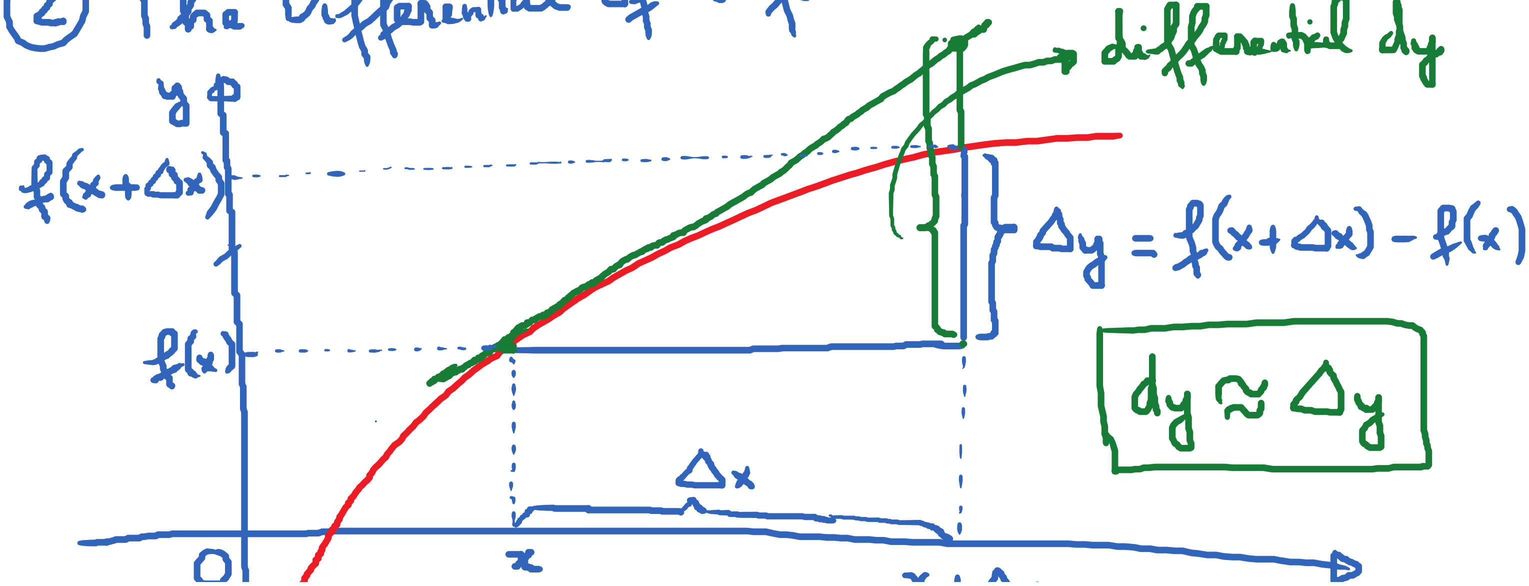
② The Differential of a function.



0
x
x + Δx
x
y = f(x)

Change in y value as x changes to $x + \Delta x$:

$$\Delta y = f(x+\Delta x) - f(x).$$

Want to estimate $\Delta y$ by $dy$.

Set $\boxed{dx = \Delta x}$ (Note: $dy \neq \Delta y$ but $dx = \Delta x$)

From picture: $\frac{\text{Rise}}{\text{Run}} = \frac{dy}{dx}$ = slope of tangent line at x

$= f'(x)$

So, $\boxed{dy = f'(x)\,dx}$

dy is called the differential of the function and it can be used to approximate $\Delta y$, the actual change of the function.

Ex. $y = f(x) = x^3 + x^2 - 2x + 1$.

Find $\Delta y$ and find $dy$ as $x$ changes from 2 to 2.05.

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= f(2.05) - f(2) = \boxed{0.71763}$$

$$dy = f'(x)\,dx = f'(2)\cdot(0.05) = 14\cdot(0.05)$$
$$= \boxed{0.7}$$

$$f'(x) = 3x^2 + 2x - 2 \longrightarrow f'(2) = 14$$

E.g. Suppose the side length of a cube is measured to be 5 cm with an error at most 0.1 cm. ($x = 5$ cm; $|dx| \leq 0.1$ (cm))

Use the measurement to calculate volume . .

$\longrightarrow$ Q: Use differential to estimate the size of the error when volume is calculated?

$$|dV| \leq ?$$

$$V(x) = x^3 \longrightarrow dV \overset{?}{=} 3x^2\,dx$$
$$= 3\cdot(5)^2\cdot dx$$

$$|dV| \leq 3\cdot 25\cdot 0.1 = 7.5\,(cm^3)$$