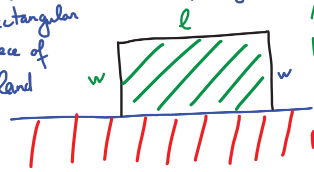
# 4.7. Optimization Problems

- $f(x)$ on a closed interval $[a,b]$
  → find abs. max / abs. min → closed interval method.
- $f(x)$ on an open interval or infinite interval
  → first derivative test. $(-\infty, \infty)$

E.g. 2400 ft of fencing

Rectangular piece of land




$A = l \cdot w$

Find $l$ and $w$ so that $A$ is maximum.





Constraints $l + 2w = 2400$ (ft)

Find $l, w$ such that $A = l \cdot w$ is maximum.

→ Turn $A$ into a function of 1 variable and use methods from calculus to find the optimal solution

$l + 2w = 2400 \longrightarrow l = 2400 - 2w$

Plug this into equation of $A$:

$\boxed{A = (2400 - 2w) \cdot w}$ → function of 1 variable $w$

$w$ is in the interval $[0, 1200]$ → find largest value of function over this closed interval.





$A(w) = (2400 - 2w)w$ on $[\boxed{0}, \boxed{1200}]$

$A(w) = 2400w - 2w^2$

$A'(w) = 2400 - 4w \longrightarrow \boxed{w = 600}$ → critical point

$A(0) = 0 \; ; \; A(1200) = 0$

$\underbrace{A(600)}_{} = 1200 \cdot 600 = \boxed{720000}$ $(ft^2)$ → largest area

if we take $w = 600$; $l = 1200$

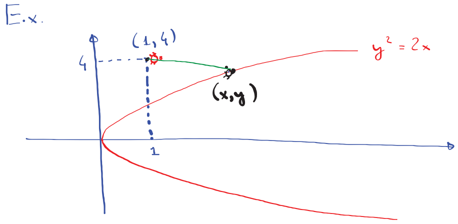




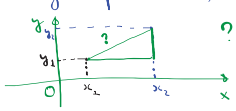
Strategy for solving optimization problems

1. Understand the problem — given / want
2. Draw a diagram
3. Introduce notations — Quantity to maximize or minimize. $Q$ / Unknowns (quantities related to $Q$)
4. Find an equation that relates $Q$ to the unknowns
5. Turn $Q$ into a function of a single variable (using the constraints on unknowns)
6. Maximize / Minimize $Q$ — closed interval method / first derivative test

Ex.




Find the point $(x,y)$ on this parabola that is closest to the given point $(1,4)$




Distance formula

$$?^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

$$? = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$





Quantity to minimize is distance D

$$D=\sqrt{(x-1)^2+(y-4)^2}$$

Find $(x,y)$ so that D is minimum.

Since $y^2=2x$, $x=\frac{y^2}{2}$

$$D=\sqrt{\boxed{\left(\frac{y^2}{2}-1\right)^2+(y-4)^2}} \quad ; \ y \text{ is in } (-\infty,\infty)$$

Trick: Instead of minimizing the expression with the square root, we minimize the one under the square root.





$$u=\left(\frac{y^2}{2}-1\right)^2+(y-4)^2$$

Find $y$ in $(-\infty,\infty)$ so that $u$ has a minimum.

$$u' = 2\left(\frac{y^2}{2}-1\right)\cdot(y) + 2(y-4)$$

$$u' = y^3-2y+2y-8 = y^3-8$$

$u'=0$ ; $y^3-8=0 \to \boxed{y=2} \longrightarrow$ critical points

$u'$: $-\infty$ (−) 2 (+) $\infty$

local min = absolute min



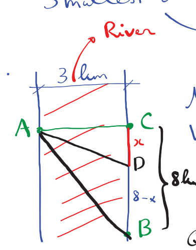


$y=2 \to 2x=4 \to x=2$.

$\longrightarrow$ $(2,2)$ is the closest point to $(1,4)$

Smallest distance: $u=5$ ; $\boxed{D=\sqrt{5}}$

Ex.




Man launches his boat from A.
Want to reach B.
He could row at a speed of 6 km/h
He could run at a speed of 8 km/h
Q: Find the point D on the other side of the river s.t. the time it takes for him to reach B is the shortest. (Find $x$ s.t. time is minimum)

$T(x)$ = total time = (time to row from A to D) + (time to run from D to B)

$$= \frac{AD}{6} + \frac{DB}{8}$$

$$T(x) = \frac{\sqrt{9+x^2}}{6} + \frac{8-x}{8} \; ; \; x \text{ is in } [0,8]$$

Find $x$ in $[0,8]$ s.t. $T$ is minimum.

$$T'(x) = \frac{x}{6\sqrt{9+x^2}} - \frac{1}{8} = 0$$

$$\frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$

$$\longrightarrow \frac{x}{\sqrt{9+x^2}} = \frac{3}{4} \longrightarrow \frac{x^2}{9+x^2} = \frac{9}{16}$$

$$\longrightarrow x^2 = \frac{9}{16}(9+x^2) \longrightarrow$$

$$x^2 = \frac{81}{16} + \frac{9}{16}x^2 \longrightarrow x^2 - \frac{9}{16}x^2 = \frac{81}{16}$$

$$\longrightarrow \frac{7}{16}x^2 = \frac{81}{16} \longrightarrow x^2 = \frac{81}{7} \longrightarrow x = \pm\frac{9}{\sqrt{7}}$$





Since $x$ is in $[0,8]$, $x = \frac{9}{\sqrt{7}} \rightarrow$ critical point

$$\left.\begin{array}{l} T(0) \\ T(8) \\ T\left(\frac{9}{\sqrt{7}}\right) \end{array}\right\}$$ whichever the smallest value is will be the minimum