4.8. L'Hopital Rule

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1 Apply L'Hopital Rule to find limits of the form $\frac{0}{0}$.

E.g. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 9} \left(\frac{0}{0}\right)$

Previously, we factor, do some algebra, cancel, etc.

L'Hopital way: $\lim_{x \to 3} \frac{2x - 2}{2x} = \frac{4}{6} = \boxed{\frac{2}{3}}$

L'Hopital Rule for $\frac{0}{0}$ limits

Suppose that we are given a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$

where $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. (a $\frac{0}{0}$ limit).

L'Hopital Rule says that:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

E.g. $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}}{1} = \frac{\frac{1}{2} + \frac{1}{2}}{1} = \boxed{1}$$

Ex. Find the given limit:

① $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$. ② $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

Sol: $\lim_{x \to 0} \frac{e^x - x - 1}{x^2} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{e^x - 1}{2x} \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$$

② $\lim_{x \to 0} \frac{\tan x - x}{x^3} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{\sec^{\boxed{2}} x - 1}{3x^2} \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2 \sec x \cdot \sec x \cdot \tan x}{6x} \left(\frac{0}{0}\right)$$

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$$\frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x}{6} \cdot \lim_{x\to 0} \frac{\tan x}{x}$$

$$\frac{1}{3}$$

$$\lim_{x\to 0} \frac{\sec^2 x}{1} = 1$$

$$= \frac{1}{3} \cdot 1 = \boxed{\frac{1}{3}}$$

$$\lim_{x\to a} \left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right) \cdot \lim_{x\to a} \left(g(x)\right)$$

L'Hopital Rule for $\frac{\infty}{\infty}$ limits

E.g. $$\lim_{x\to\infty} \frac{2x^2+5}{5x^2+x+10} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{4x}{10x+1} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{4}{10} = \boxed{\frac{2}{5}}$$

E.g. $\lim_{x\to 0} \dfrac{e^x}{1-\cos x}$ $\left(\dfrac{1}{0}\right) = \infty$

$1-\cos x \to 0$

x	$\dfrac{e^x}{1-\cos x}$
0.1	
0.01	
0.001	
0.0001	

(Cannot use L'Hopital Rule)

Ex. Find the given limit.

① $\lim_{x\to\infty} \dfrac{\ln(5x+3)}{\ln(2x+5)} \to 1$

② $\lim_{x\to 0^+} \dfrac{e^x - 1 - x}{x \sin x} \to \dfrac{1}{2}$

③ $\lim_{x\to\infty} \dfrac{\ln x}{x} \to 0$

$\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are called indeterminate form.

There are other indeterminate forms:

$$\boxed{\infty - \infty,\quad 0\cdot\infty,\quad \infty^0,\ 1^\infty,\ 0^0}$$

You cannot apply L'Hopital Rule directly here

→ When you get one of those forms, you need to use algebraic manipulations to turn them back to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then apply L'Hopital Rule.

E.g. Find $\lim_{x\to\infty} \underbrace{x}_{\infty} \cdot \underbrace{\tan\left(\frac{1}{x}\right)}_{0}$ ($0\cdot\infty$ form)

$$= \lim_{x\to\infty} \boxed{\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}} \quad \begin{matrix}\to 0 \\ \to 0\end{matrix} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \sec^2\left(\underbrace{\frac{1}{x}}_{\to 0}\right) = \sec^2(0) = \boxed{1}$$

$0\cdot\infty = 1$

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Ex. Find $\lim_{x\to\infty} \boxed{x^3} \cdot \boxed{e^{-x^2}}$

$\infty \cdot 0$

$$= \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \quad \left(\frac{\infty}{\infty}\right) \longrightarrow \text{L'Hopital Rule}$$

$$= \lim_{x\to\infty} \frac{3x^2}{e^{x^2}\cdot 2x} = \lim_{x\to\infty} \frac{3x}{e^{x^2}2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{3}{e^{x^2}\cdot 2x\cdot 2} = \lim_{x\to\infty} \frac{3}{4x\cdot e^{x^2}}$$

$$= \frac{3}{\infty} = \boxed{0} \longleftrightarrow \infty \cdot 0$$

<u>Note:</u> $\lim_{x\to a} f(x)\cdot g(x)$ where $\lim_{x\to a} f(x) = \infty$

and $\lim_{x\to a} g(x) = 0$

$\infty \cdot 0$ form

2 ways to proceed

$$\lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \quad \frac{\infty}{\infty}$$

$$\lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}} \quad \frac{0}{0}$$

then use L'Hopital

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* How to deal with ∞^0, 1^∞, 0^0 ?

E.g. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ (1^∞)

let $L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Take ln of both sides:

$$\ln(L) = \lim_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^x$$

$\infty \cdot 0$

$$\ln(L) = \boxed{\lim_{x \to \infty} x \cdot \ln\left(1 + \frac{1}{x}\right)}$$

$\infty \cdot 0$ form

$$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{1}{x}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \qquad 1^{\infty}$$

$$e^{\ln(L)} = e^{1} \longrightarrow L = \boxed{e}$$

$\infty - \infty$

E.g. $\lim_{x\to\infty}\left(\overset{\infty}{x} - \overset{\infty}{\sqrt{x^2+3x}}\right)$

$$= \lim_{x\to\infty} \frac{x - \sqrt{x^2+3x}}{1} \cdot \frac{x+\sqrt{x^2+3x}}{x+\sqrt{x^2+3x}}$$

$$= \lim_{x\to\infty} \frac{x^2 - (x^2+3x)}{x+\sqrt{x^2+3x}} = \lim_{x\to\infty} \frac{-3x}{x+\sqrt{x^2+3x}} \quad \left(\frac{-\infty}{\infty}\right)$$

→ L'Hopital.

$$= \lim_{x\to\infty} \frac{-3}{1 + \frac{2x+3}{2\sqrt{x^2+3x}}} = \boxed{-\frac{3}{2}}$$

$\frac{2x+3}{2\sqrt{x^2+3x}} \to 1$