4.9. Newton's Method

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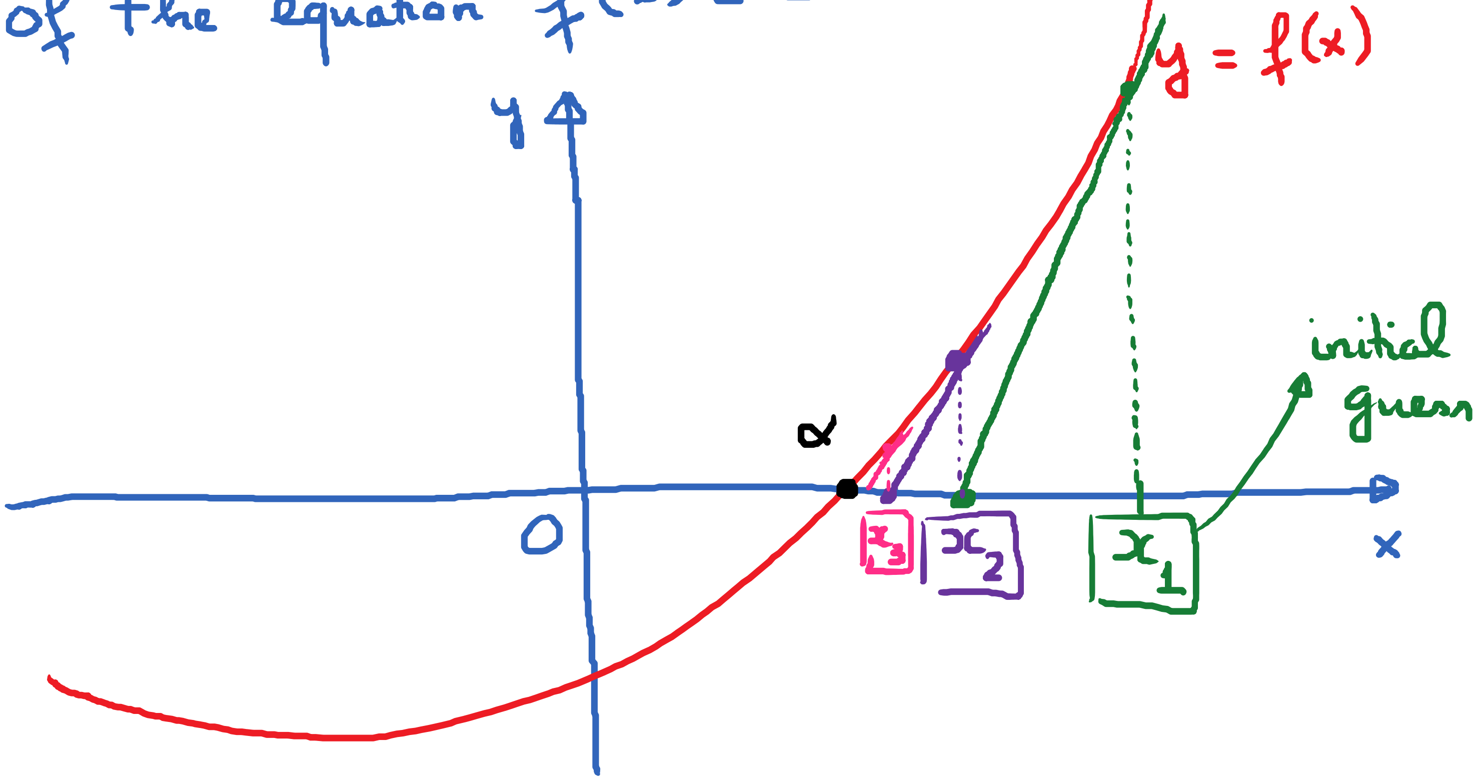
Newton's method is used to estimate the solution of the equation $f(x) = 0$.



α is a solution. (intersection btwn f and x-axis)

Goal: estimate α

x_2 = intersection between the x-axis and the tangent line to graph of $y = f(x)$ at $(x_1, f(x_1))$

To find x_2.

Step 1: Find the equation of the tangent line to graph of $y = f(x)$ at $(x_1, f(x_1))$

Slope: $f'(x_1)$. Point: $(x_1, f(x_1))$

Point - Slope Equation:

$$y - f(x_1) = f'(x_1)\cdot(x - x_1)$$

→ Slope intercept equation:

$$\boxed{y = f'(x_1)\cdot(x - x_1) + f(x_1)}$$

Step 2: Find x_2.

Set $y = 0$ in the above equation and solve for x

$$f'(x_1)(\boxed{x} - x_1) + f(x_1) = 0$$

$$f'(x_1)\cdot(x - x_1) = -f(x_1)$$

$$x - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

So, $\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$

Similarly, $\boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}}$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This is the formula for Newton's method.

The formula tells you how to find the next approximation to α based on the guess immediately before it.

E.g. Use Newton's method with initial guess $x_1 = 2$ to estimate the solution (root, zero) of the equation:

$$\underbrace{x^3 + 5x - 1}_{f(x)} = 0$$

$f'(x) = 3x^2 + 5$

Find x_2, x_3, x_4.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{17}{17}$$

$$\boxed{x_2 = 1}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1 - \frac{5}{8} = \frac{3}{8} \simeq 0.375$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{3}{8} - \frac{0.9277}{5.42} \simeq \boxed{0.204}$$

Ex. Use Newton's method to approximate $\sqrt{2}$.

The number $\sqrt{2}$ is the positive solution to the equation: $\underbrace{x^2 - 2}_{f(x)} = 0$

① Find the formula that relates x_{n+1} to x_n in Newton's method for this function.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= \frac{x_n \cdot 2x_n}{1 \cdot 2x_n} - \frac{x_n^2 - 2}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2}{2x_n} - \frac{x_n^2 - 2}{2x_n}$$

$$\boxed{x_{n+1} = \frac{x_n^2 + 2}{2x_n}} \rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$\boxed{x_{n+1} = \frac{1}{2} \cdot \left(x_n + \frac{2}{x_n} \right)}$$

$x_1 = 3$; $x_2 = \frac{1}{2}\left(3 + \frac{2}{3}\right) = \frac{11}{6} \approx 1.8333$

$x_3 = 1.4621$; x_4 ...