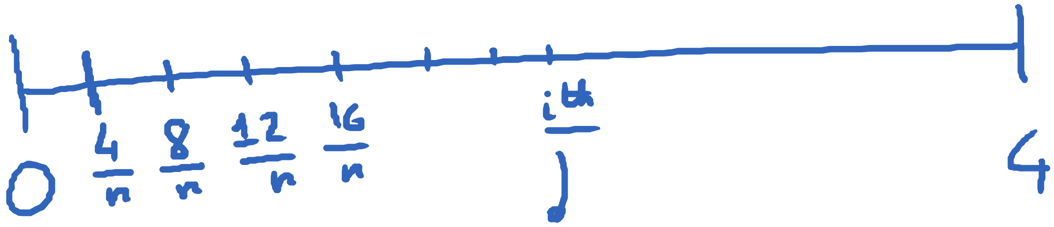
Take any n. width of a rectangle: $\frac{4}{n}$.



$$\underbrace{\frac{4}{n}\cdot i}_{x_i}$$

1st right endpoint: $\frac{4}{n}$

2nd right endpoint: $\frac{8}{n} = \frac{4}{n}\cdot 2$

3rd right endpoint: $\frac{12}{n} = \frac{4}{n}\cdot 3$

$\vdots$

ith right endpoint: $\frac{4}{n}\cdot i$

$$R_n = \frac{4}{n}\sum_{i=1}^{n} f(x_i) = \frac{4}{n}\cdot\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)$$

$$R_n = \frac{4}{n}\cdot\sum_{i=1}^{n}\left(\frac{4i}{n}\right)^2 = \frac{4}{n}\cdot\sum_{i=1}^{n}\frac{16i^2}{n^2}$$

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$$R_n = \frac{4}{n} \cdot \frac{16}{n^2} \cdot \sum_{i=1}^{n} i^2 \longrightarrow \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{4}{\cancel{n}} \cdot \frac{16}{n^2} \cdot \frac{\cancel{n}(n+1)\cdot(2n+1)}{6}$$

$$R_n = \frac{32(n+1)(2n+1)}{3n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{32(n+1)(2n+1)}{3n^2} = \boxed{\frac{64}{3}}$$

Exact Area

Idea of Riemann Sum.

Use L_n, R_n to find the area under $y = f(x)$ on the interval $[a,b]$.

1 Divide the interval $[a,b]$ into n subintervals.

length of each subinterval $= \frac{b-a}{n} = \Delta x$

(width of each small rectangle)

② <u>Right Riemann Sum</u>:

i^{th} right end point is : $a + i \cdot \Delta x = x_i$

$$R_n = \Delta x \cdot \sum_{i=1}^{n} f(x_i) = \Delta x \cdot \sum_{i=1}^{n} f(a + i\Delta x)$$

<u>Left Riemann Sum</u>:

i^{th} left endpoint is : $a + (i-1)\Delta x$.

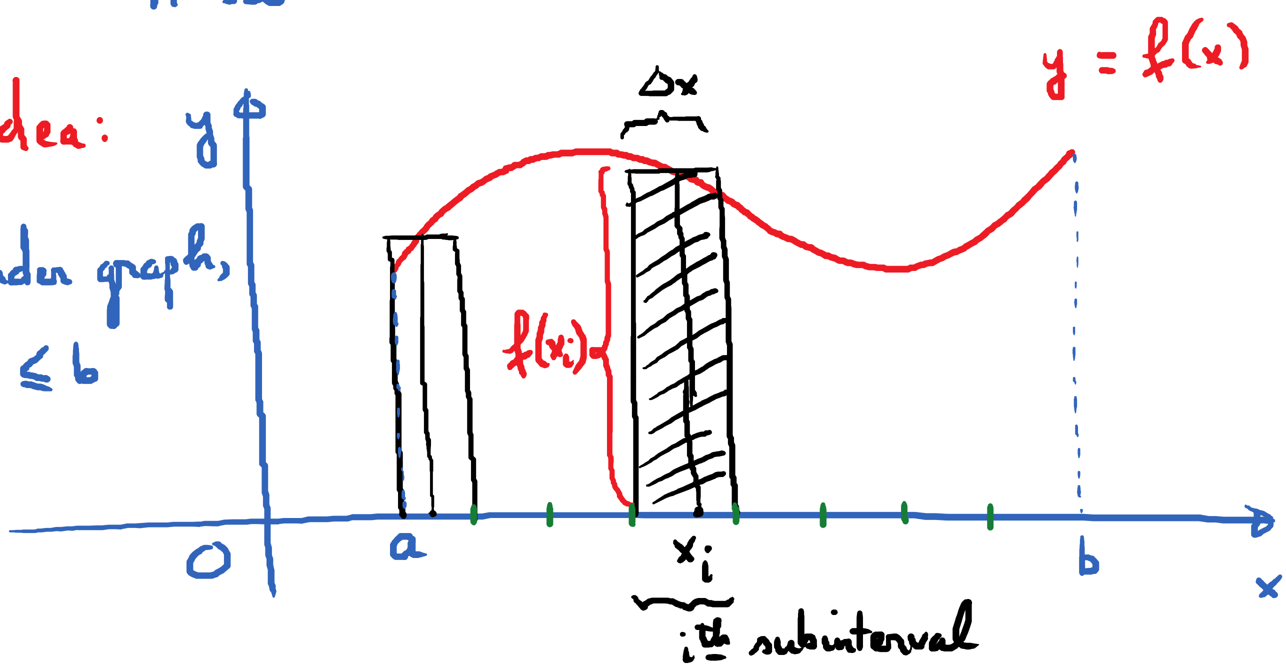
$$L_n = \Delta x \cdot \sum_{i=1}^{n} f(x_{i-1}) = \Delta x \cdot \sum_{i=1}^{n} f(a + (i-1)\Delta x)$$

③ $\text{Area} = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n$

Key idea:

Area under graph,

$a \leq x \leq b$



$$\lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{n} \underbrace{f(x_i)\cdot\Delta x}_{\text{Area of } i^{\underline{th}} \text{ rectangle}} = \text{Exact area under } y = f(x),\ a \le x \le b$$

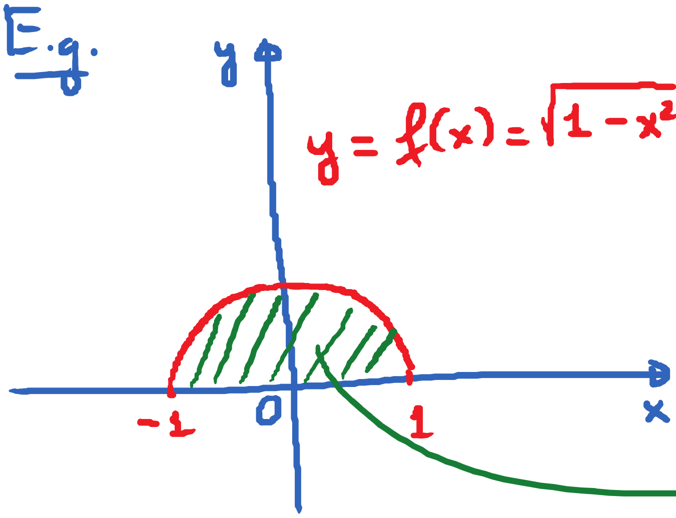
$$\left(\text{Here } \Delta x = \frac{b-a}{n}\right)$$

Important Notation:

This limit is called the definite integral of $f(x)$ on $[a,b]$. It is denoted by:

$$\int_a^b f(x)\,dx = \text{exact area under } y = f(x) \text{ on } [a,b]$$

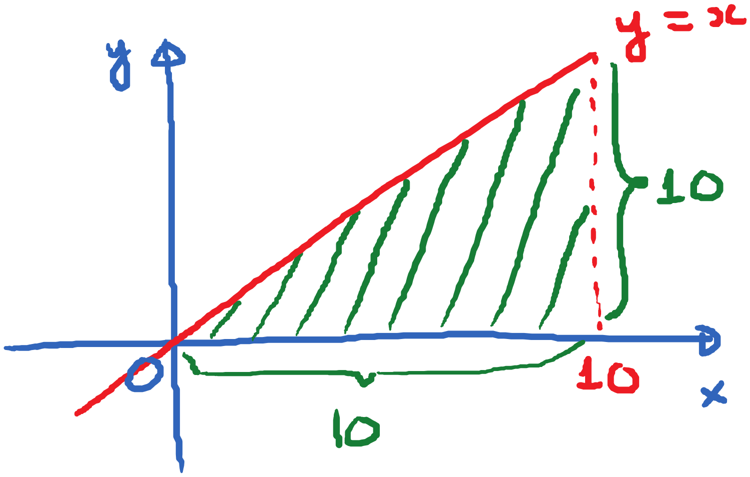
E.g.



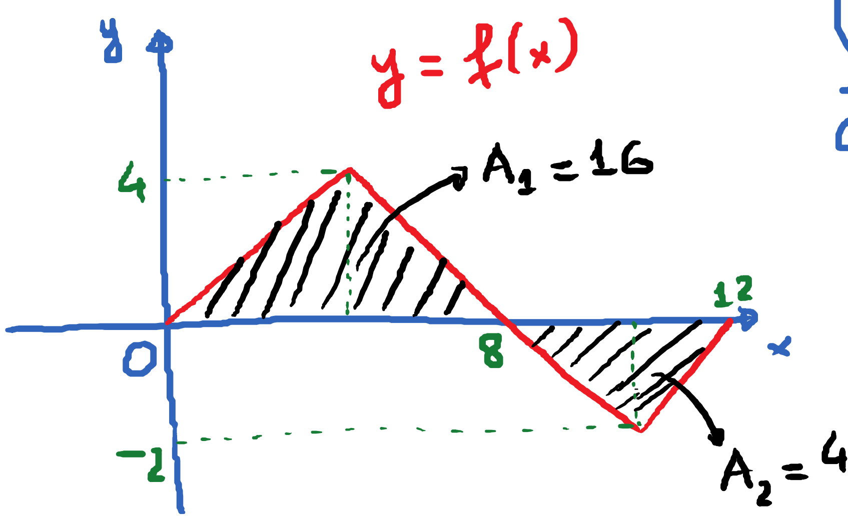
Find: $\int_{-1}^{1} f(x)\,dx$?

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \text{Area} = \frac{\pi}{2}$$

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$



$$\int_{0}^{10} x\,dx \stackrel{?}{=} \text{Area} = 50$$



$$\int_{0}^{12} f(x)\,dx = A_1 - A_2$$

$$= 16 - 4 = 12.$$

Some useful properties of the definite integral:

1. $$\int_a^a f(x)\,dx = 0$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3. $$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

4. $$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

5. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$