The Fundamental Theorem of Calculus

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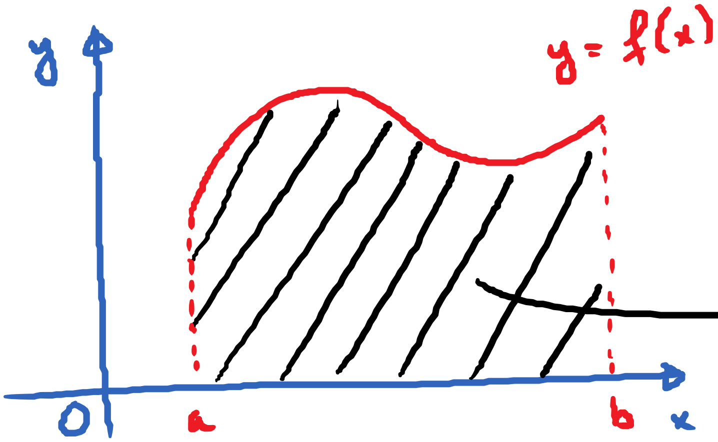
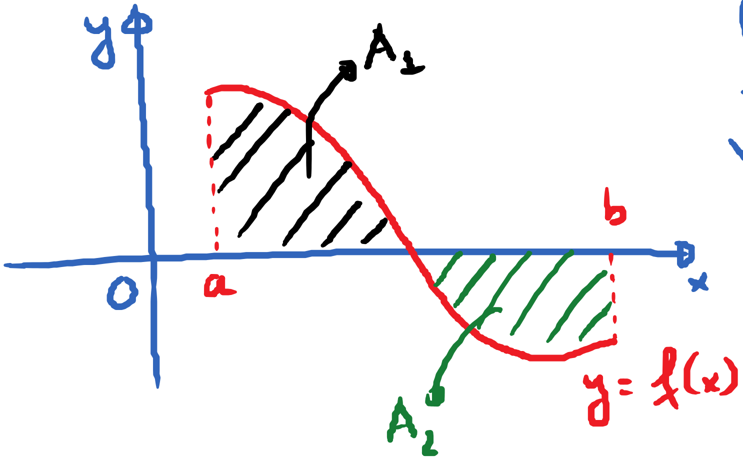


$$\int_a^b f(x)\,dx = \text{Area}$$



$$\underbrace{\int_a^b f(x)\,dx}_{} = A_1 - A_2$$

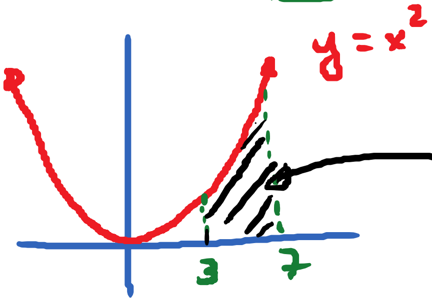
read as the definite integral of $f(x)$ with respect to x from a to b.

The Fundamental Theorem of Calculus Part I.

E.g. $\displaystyle\int_{\boxed{3}}^{\boxed{7}} x^2\,dx$

Step 1: Find antiderivative:

$$F(x) = \int x^2\,dx = \frac{x^3}{3}$$



Step 2: $F(7) - F(3)$

$$= \frac{343}{3} - \frac{27}{3} = \boxed{\frac{316}{3}}$$

FTC - II

If f is a continuous function on $[a,b]$ and $F(x)$ is an antiderivative of $f(x)$; i.e., $F'(x)=f(x)$, then:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Useful notation: $F(x)\Big|_a^b = F(b) - F(a)$

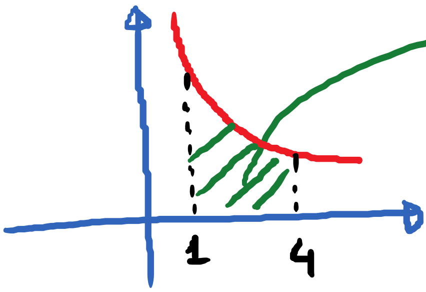
$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

E.g.

$$\int_1^4 \frac{4}{x^2}\,dx = 4\cdot\int_1^4 \frac{1}{x^2}\,dx = 4\cdot\int_1^4 x^{-2}\,dx$$

$$= 4\cdot\frac{x^{-2+1}}{-2+1}\Big|_1^4 = -4\cdot\frac{1}{x}\Big|_1^4$$

$$-4\left(\frac{1}{4}-\frac{1}{1}\right) = -4\cdot\left(-\frac{3}{4}\right) = \boxed{3}$$



Ex. Evaluate the given definite integrals.

(1) $\displaystyle\int_4^8 \left(4t^{\frac{5}{2}} - 3t^{\frac{3}{2}}\right)dt$

(2) $\displaystyle\int_1^4 \frac{2-\sqrt{u}}{u^2}\,du$

(3) $\displaystyle\int_0^{\pi/2} \sin(\theta)\,d\theta$

(4) $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$

* (5) $\displaystyle\int_0^5 \sqrt{25-x^2}\,dx$. (Hint: don't try to find the antiderivative)

(3) $$\int_0^{\pi/2} \sin(\theta)\,d\theta = -\cos(\theta)\Big|_0^{\pi/2} = -\left[\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - \underbrace{\cos(0)}_{1}\right]$$
$$= -(-1) = 1.$$

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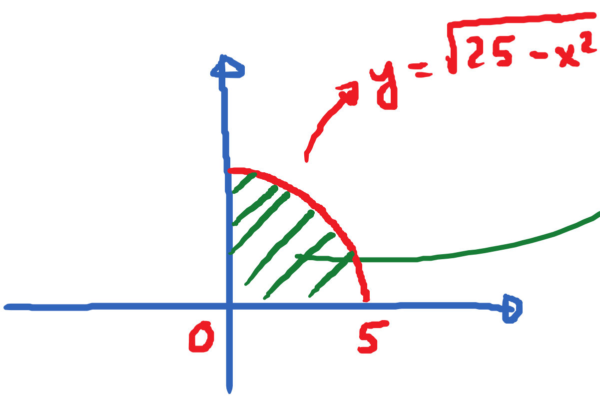
$$\int_{\pi/4}^{\pi/2} \csc^2\theta \, d\theta = -\cot(\theta)\Big|_{\pi/4}^{\pi/2}$$

$$= -\left[\underbrace{\cot\left(\frac{\pi}{2}\right)}_{0} - \underbrace{\cot\left(\frac{\pi}{4}\right)}_{1}\right]$$

$$= -(-1) = 1$$

$$\int_0^5 \sqrt{25-x^2}\, dx = \frac{25\pi}{4}.$$



FTC, Part I.

<u>E.g.</u> $f(t) = t^2$. on $[1,3]$

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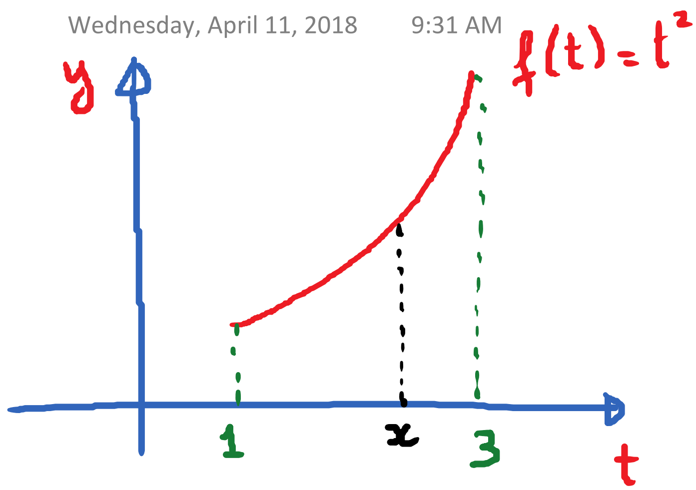

$$\int_1^x f(t)\,dt = \int_1^x t^2\,dt$$

$$= \frac{t^3}{3}\Big|_1^x$$

$$= \frac{x^3}{3} - \frac{1}{3}$$

$$\int_1^x f(t)\,dt = \frac{x^3}{3} - \frac{1}{3}. \qquad (f(t)=t^2)$$

$$\frac{d}{dx}\left(\int_1^x \boxed{f(t)}\,dt\right) = \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1}{3}\right) = x^2 = f(x)$$

$$\longrightarrow \frac{d}{dx}\left(\int_1^x f(t)\,dt\right) = f(x)$$

E.g. $\frac{d}{dx}\left(\int_{100}^x e^{-t^2}\,dt\right) = e^{-x^2}.$

FTC, part I

$$\underbrace{\frac{d}{dx}}_{\text{take derivative w.r.t. } x \text{ of that function}} \left(\underbrace{\int_a^x f(t)\,dt}_{\text{function of } x} \right) = f(x)$$

E.g. $\dfrac{d}{dr}\left(\displaystyle\int_4^r \dfrac{d\theta}{\sqrt{16-\theta^2}} \right) = \dfrac{1}{\sqrt{16-r^2}}$

E.g. $\dfrac{d}{dx}\left(\displaystyle\int_1^x \sec(t)\,dt \right) = \sec(x)$

$\dfrac{d}{dx}\left(\displaystyle\int_1^{x^4} \sec(t)\,dt \right) = \left[\sec(x^4)\right] \cdot (4x^3)$

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = f(g(x)) \cdot g'(x)$$

(by the Chain Rule)

Ex. Find the derivative of the given function:

1. $g(x) = \displaystyle\int_1^x \frac{1}{t^3+1}\,dt$

2. $h(x) = \displaystyle\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$

3. $w(x) = \displaystyle\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$

4. $a(x) = \displaystyle\int_{\sqrt{x}}^{2x} \arctan(t)\,dt$

Ex.

$$g(x) = \int_0^x (1-t^2)e^{t^2}\,dt.$$

Q: On what interval is g decreasing?