5.5. Integration by Substitution

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<u>Recall:</u> FTC.

FTC - Part II :

$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = F(x)\Big|_a^b = F(b) - F(a)$$

FTC - Part II :

$$\underbrace{\frac{d}{dx}\left(\int_a^{\boxed{x}} \boxed{f(t)}\,dt\right)}_{\text{result is a function of } x} = f(x)$$

More generally, $\displaystyle \frac{d}{dx}\left(\int_a^{\boxed{g(x)}} f(t)\,dt\right) = f(g(x))\cdot g'(x)$

Ex. Evaluate the integral

① $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx$

② $\displaystyle\int_0^1 \left(u^e + e^u\right) du$

③ $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz$

④ $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8\,du}{1+u^2}$

Ex.

Find $h'(x)$ given $\displaystyle h(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^4+1}\,dt$

Sol:

① $\displaystyle\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx = \int_1^9 \left(x^{1/2} - x^{-1/2}\right) dx$

$$= \underbrace{\left(\frac{2x^{3/2}}{3} - 2x^{1/2}\right)}_{F(x)\text{ - antiderivative}}\Bigg|_1^9 = \left[\frac{2\cdot(9)^{3/2}}{3} - 2\cdot(9)^{1/2}\right] - \left[\frac{2(1)^{3/2}}{3} - 2\cdot(1)^{1/2}\right]$$

$$[18-6]-\left[\frac{2}{3}-2\right] = 12-\left(-\frac{4}{3}\right) = \boxed{\frac{40}{3}}.$$

② $$\int_0^1 \left(u^e + e^u\right) du = \underbrace{\left(\frac{u^{e+1}}{e+1} + e^u\right)}_{F(x)}\Bigg|_0^1$$

$$= \left[\frac{(1)^{e+1}}{e+1} + e^1\right] - \left[\frac{(0)^{e+1}}{e+1} + e^0\right]$$

$$= \boxed{\frac{1}{e+1} + e - 1}$$

③ $$\int_1^{18} \sqrt{\frac{3}{z}}\, dz$$

$$= \int_1^{18} \left(\frac{3}{z}\right)^{1/2} dz = \int_1^{18} \frac{\boxed{3^{1/2}}}{z^{1/2}}\, dz$$

$$= 3^{1/2} \cdot \int_1^{18} z^{-1/2}\, dz = \underbrace{3^{1/2} \cdot \frac{z^{1/2}}{1/2}}_{F(x)}\Bigg|_1^{18}$$

$$= 2\sqrt{3} \cdot z^{1/2}\Big|_1^{18} = 2\sqrt{3}\left((18)^{1/2} - 1\right)$$

(4) $$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8\,du}{1+u^2} = 8 \cdot \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+u^2}\,du$$

$$= \underbrace{8\arctan(u)}_{F(x)} \Big|_{1/\sqrt{3}}^{\sqrt{3}} = 8\left[\arctan(\sqrt{3}) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 8\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = 8 \cdot \frac{\pi}{6} = \boxed{\frac{4\pi}{3}}.$$

Ex. $h(x) = \int_1^{\boxed{\sqrt{x}}} \frac{t^2}{t^4+1}\,dt$; $h'(x) \stackrel{?}{=}$

$$h'(x) = \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} \cdot \frac{1}{2\sqrt{x}} = \frac{x}{x^2+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{x}{2\sqrt{x}(x^2+1)}$$

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Integration By Substitution (u-Sub)

$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	$\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$
$\int \dfrac{dx}{x} = \ln\lvert x\rvert + C$	$\int \dfrac{du}{u} = \ln\lvert u\rvert + C$
$\int \sin(x)\,dx = -\cos x + C$	$\int \sin(u)\,du = -\cos(u) + C$
$\int \cos(x)\,dx = \sin x + C$	$\int \cos(u)\,du = \sin(u) + C$
$\int e^x\,dx = e^x + C$	$\int e^u\,du = e^u + C$
$\int \dfrac{dx}{1+x^2} = \arctan(x) + C$	$\int \dfrac{du}{1+u^2} = \arctan(u) + C$

$$\int \sin\left(\boxed{x^3}\right)\cdot\boxed{3x^2}\,dx.$$

Let $\underbrace{u = x^3}$

$$du = 3x^2\,dx$$

$$\int \sin(u)\cdot\underbrace{\boxed{3x^2\,dx}}_{du}$$

differential

$$d(f(x)) = f'(x)\,dx$$

The original integral (in x) now becomes the following integral in u:

$$\int \sin(u)\,du = -\cos(u) + C$$

$$= -\cos(x^3) + C.$$

$$\int \sin(x^3)\cdot \underbrace{\boxed{6x^2\,dx}}_{2\,du} \stackrel{?}{=}$$

Let $u = x^3$

$du = 3x^2\,dx$

$$\int \sin(u)\cdot 2du = 2\int \sin(u)\,du = -2\cos(u) + C.$$

$$= -2\cos(x^3) + C.$$

$$\int \sin(\underbrace{\boxed{x^3}}_{u})\cdot \underbrace{18x^2\,dx}_{6\,du} = -6\cos(x^3) + C.$$

Find $\frac{1}{3}\int \sin(x^3)\, 3x^2\, dx$. Let $u = x^3$

$du = 3x^2\,dx$

$$\frac{1}{3}\int \sin(u)\,du = -\frac{1}{3}\cos(u) + C$$

$$= -\frac{1}{3}\cos(x^3) + C.$$

In general, if you have an integral of the form

$$\int f(\overset{u}{\boxed{g(x)}}) \cdot \boxed{g'(x)\,dx}$$

Then u-sub will work.

Let $u = g(x)$. $du = g'(x)\,dx$

Original integral becomes $\int f(u)\,du$

$$\frac{1}{2}\int e^{\overset{u}{\boxed{x^2+4}}}\, \underbrace{2x\,dx}_{du} \stackrel{?}{=}$$

Let $u = x^2+4$.

$du = 2x\,dx$

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}\cdot e^u + C = \boxed{\frac{1}{2}e^{x^2+4} + C}$$

E.g. $-\int \underbrace{\boxed{\cos^2(x)}}_{u^2} \underbrace{(-\sin x\, dx)}_{du}$ Let $u = \cos(x)$

$du = -\sin(x) \cdot dx$

$$-\int u^2\, du = -\frac{u^3}{3} + C = \boxed{-\frac{\cos^3(x)}{3} + C}$$

Ex. $\int \frac{(\ln x)^2}{x}\, dx$; $u = \ln x$; $du = \frac{1}{x}\, dx$

du

$$\int u^2\, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

Ex. ① $\int x^2 \cdot e^{x^3}\, dx$ ② $\int e^x \cos(e^x)\, dx$

③ $\int \frac{\sin(t)}{\cos^3(t)}\, dt$ ④ $\int y\sqrt{y^2 - 5}\, dy$

⑤ $\int \sqrt{\cot x} \cdot \csc^2 x\, dx$ ⑥ $\int \frac{\tan^{-1} x}{1 + x^2}\, dx$