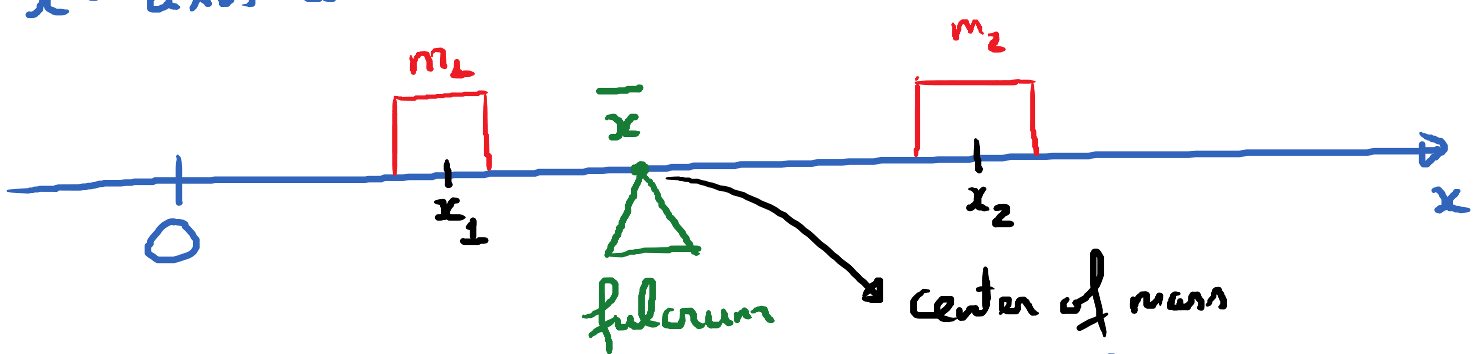
2.6. Moments and Center of Mass

Tuesday, February 6, 2018 1:08 PM

* 2 point masses m_1 and m_2 located on the x-axis at the x-coordinates x_1 and x_2.

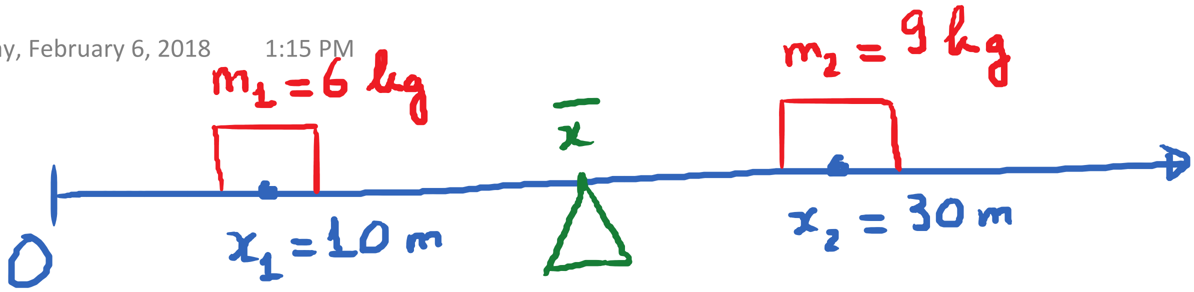


The center of mass $\bar{x}$ is the point on the x-axis where we should place the fulcrum to make the system balance.

The formula for $\bar{x}$ is the "weighted average" of x_1 and x_2

$$\boxed{\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

E.g.



Find $\bar{x}$?

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{6 \cdot 10 + 9 \cdot 30}{6 + 9} = 22 \text{ (m)}$$

In general, if we have n masses $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$ on the x-axis, the center of mass $\bar{x}$ is given by:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$$

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}}$$

moment of the system

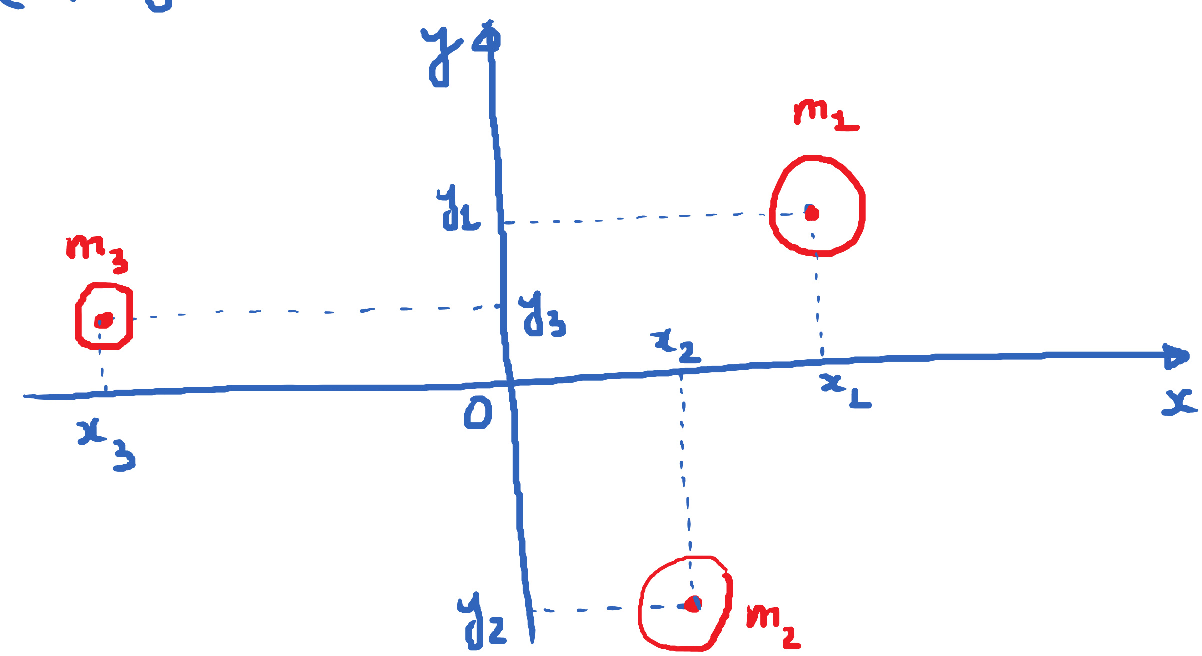
total mass of the system

Note: The quantity

$$M = \sum_{i=1}^{n} m_i x_i$$ is called the moment of the system.

$$\text{center of mass} = \frac{\text{moment}}{\text{total mass}}$$

Now, consider a system of n masses located at n points on the xy-plane: (x_1, y_1); (x_2, y_2); … (x_n, y_n)



Picture for n = 3 masses

The center of mass is the point $(\bar{x}, \bar{y})$ where the coordinates $\bar{x}$ and $\bar{y}$ are given by:

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \; ; \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}}$$

$m = \sum_{i=1}^{n} m_i$ is the total mass of the system.

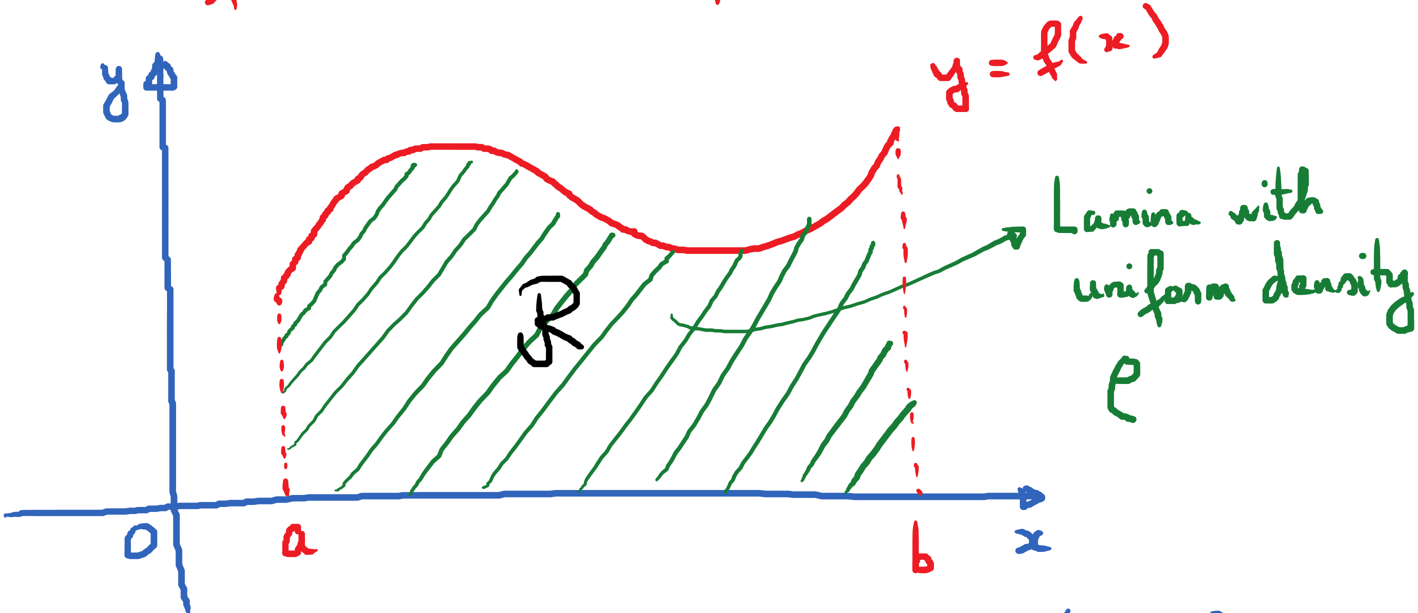
$M_y = \sum_{i=1}^{n} m_i x_i$ is the y-moment or the moment about the y-axis

$M_x = \sum_{i=1}^{n} m_i y_i$ is the x-moment or the moment about the x-axis.

$$\longrightarrow \boxed{\bar{x} = \frac{M_y}{m} \; ; \quad \bar{y} = \frac{M_x}{m}}$$

$$\bar{x} = \frac{y\text{-moment}}{\text{mass}} \; ; \quad \bar{y} = \frac{x\text{-moment}}{\text{mass}}$$

* Center of mass of thin plates (thin plate = lamina)



Q: How do we find the $\underbrace{\text{center of mass}}_{\text{Centroid}}$ $(\bar{x}, \bar{y})$ of this lamina?

Moment of R about y-axis:

$$\boxed{M_y = \rho \cdot \int_a^b x\, f(x)\, dx}$$

Moment of R about x-axis:

$$\boxed{M_x = \rho \int_a^b \frac{1}{2} [f(x)]^2\, dx}$$

Mass of R : $m = \rho \cdot \int_a^b f(x)\,dx$

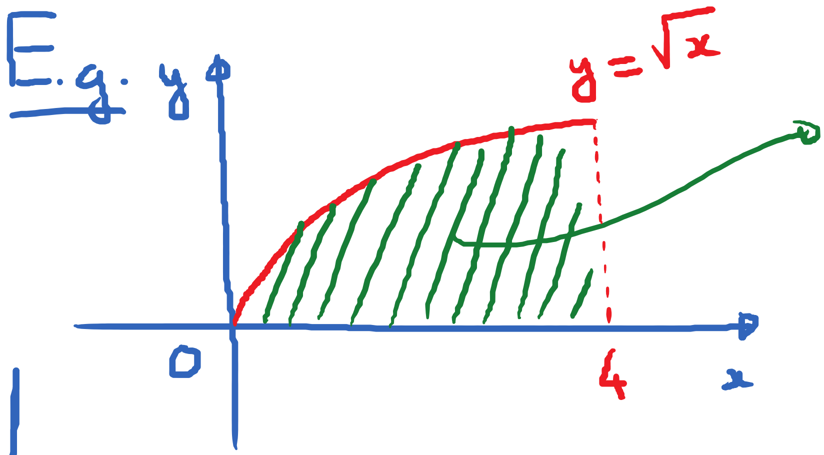
Centroid $(\bar{x}, \bar{y})$ of R is :

$$\bar{x} = \frac{M_y}{m} = \frac{\cancel{\rho} \cdot \int_a^b x f(x)\,dx}{\cancel{\rho} \int_a^b f(x)\,dx}$$

$$\boxed{\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}}$$

$$\boxed{\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}}$$

E.g.



lamina with uniform density ρ.

Q: Find the centroid $(\bar{x}, \bar{y})$

$$m = \int_0^4 \sqrt{x}\, dx = \int_0^4 x^{\frac{1}{2}} dx = \frac{2x^{\frac{3}{2}}}{3}\Big|_0^4 = \frac{2}{3}\cdot(4)^{\frac{3}{2}}$$

$$= \frac{2}{3}\cdot(\sqrt{4})^3 = \frac{16}{3}$$

$$M_y = \int_0^4 x\cdot\sqrt{x}\, dx = \int_0^4 x\cdot x^{\frac{1}{2}} dx = \int_0^4 x^{3/2} dx$$

$$= \frac{2x^{5/2}}{5}\Big|_0^4 = \frac{2}{5}\cdot(4)^{5/2} = \frac{64}{5}$$

$$M_x = \int_0^4 \frac{1}{2}\cdot(\sqrt{x})^2 dx = \int_0^4 \frac{1}{2} x\, dx = \frac{x^2}{4}\Big|_0^4 = 4$$

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \boxed{\frac{12}{5}}. \qquad \bar{y} = \frac{M_x}{m} = \frac{4}{16/3} = \boxed{\frac{3}{4}}$$