Ex. (1) $\int \sin^2 x \, dx$ (2) $\int \sin^2 x \cos^2 x \, dx$

Ans: (1) $\frac{x}{2} - \frac{\sin(2x)}{4} + C$

(2) $\frac{x}{8} - \frac{\sin(4x)}{32} + C$

Form: $\int \tan^n x \sec^m x \, dx$

Key formulas:

$\frac{d}{dx}(\tan x) = \sec^2 x$ ; $\frac{d}{dx}(\sec x) = \sec x \tan x$.

$\int \sec^2 x \, dx = \tan x + C$. $\int \sec x \tan x \, dx = \sec x + C$

$\int \tan x \, dx = \ln|\sec x| + C$.

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

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Case 1: m, the power of secant, is even.

$u^6$

E.g. $\int \tan^6 x \cdot \sec^4 x \, dx$

let $u = \tan x$ ; $du = \sec^2 x \, dx$

$$\int \underbrace{\tan^6 x}_{u^6} \, \underbrace{\sec^2 x}_{(1+u^2)} \, \overbrace{\sec^2 x \, dx}^{du}$$

$$\int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \boxed{\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C}$$

Strategy for power of secant is even:

① Save a factor $\sec^2 x\, dx$

② $u = \tan x$

Case 2: $n$, the power of tangent is odd and $m \geq 1$.

E.g. $\int \tan^5 x \sec^7 x \, dx$

$$= \int \underbrace{\left(\tan^4 x\right)}_{(u^2-1)^2} \underbrace{\left(\sec^6 x\right)}_{u^6} \underbrace{\left(\sec x \tan x \, dx\right)}_{du}$$

let $u = \sec x$ ; $du = \sec x \tan x \, dx$

$\tan^2 x = \sec^2 x - 1$ . $\tan^4 x = \left(\tan^2 x\right)^2$

$$= \int \left(u^2 - 1\right)^2 \cdot u^6 \, du = \int \left(u^4 - 2u^2 + 1\right) \cdot u^6 \, du$$

$$= \int \left(u^{10} - 2u^8 + u^6\right) du = \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C$$

Strategy: power of tangent is odd, has at least a secant factor.

1. Save a factor of $\sec x \tan x \, dx$
2. let $u = \sec x$.

Case 3: Neither 1 nor 2.

This could require a variety of tricks: like integrations by parts, substitution, trig identities, etc.

E.g. $\int \sec^3 x \, dx$.

Hint: Integration by parts.

$$\begin{cases} u = \sec x \\ dv = \sec^2 x \, dx \end{cases}$$

**Form:**

$$\int \sin(nx)\cos(mx)\,dx$$

$$\int \sin(nx)\sin(mx)\,dx$$

$$\int \cos(nx)\cos(mx)\,dx$$

**Strategy:** Product-to-Sum Identities.

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

**E.g.**

$$\int \underbrace{\cos(6x)\cos(3x)}\,dx$$

$$= \int \frac{1}{2}\left[\cos(3x) + \cos(9x)\right]dx$$

$$\frac{1}{2}\int\left(\cos(3x) + \cos(9x)\right)dx$$

$$= \frac{1}{2}\left[\frac{\sin(3x)}{3} + \frac{\sin(9x)}{9}\right] + C$$

$$= \boxed{\frac{\sin(3x)}{6} + \frac{\sin(9x)}{18} + C}$$