3.3. Trigonometric Substitution

Tuesday, February 20, 2018 1:18 PM

Goal: Find integrals when the integrands have radical expressions in them by using trig substitution.

Strategy for trig substitution

Expression	Trig Sub	Identity
$\sqrt{a^2 - x^2}$	$\boxed{x = a\sin\theta}$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 - (a\sin\theta)^2}$ $= \sqrt{a^2 - a^2\sin^2\theta}$ $= \sqrt{a^2\cdot(1-\sin^2\theta)}$ $= \sqrt{a^2\cdot\cos^2\theta}$ $= a\cos\theta$
$\sqrt{a^2 + x^2}$	$\boxed{x = a\tan\theta}$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sqrt{a^2 + a^2\tan^2\theta}$ $= \sqrt{a^2(1+\tan^2\theta)}$ $= \sqrt{a^2\sec^2\theta}$ $= a\sec\theta$

$\sqrt{x^2 - a^2}$

$\boxed{x = a\sec\theta}$

$0 \leq \theta < \frac{\pi}{2}$ or

$\pi \leq \theta < \frac{3\pi}{2}$

$$\sqrt{a^2\sec^2\theta - a^2}$$
$$= \sqrt{a^2(\sec^2\theta - 1)}$$
$$= \sqrt{a^2\tan^2\theta}$$
$$= a\tan\theta$$

Point of a trig sub: * turn the integral into an integral of trig functions & no more radicals

→ evaluate this new integral in terms of θ

→ Use geometry and trig identities to convert the result back to an expression in x.

E.g. Find $\int \sqrt{9 - x^2}\, dx$

Trig sub: $x = 3\sin\theta$; $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$

$dx = 3\cos\theta\, d\theta$

$$\int \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta \, d\theta \, .$$

$9(1-\sin^2\theta)$

$= 9\cos^2\theta$

$$\int \sqrt{9\cos^2\theta} \cdot 3\cos\theta \, d\theta$$

$$\int 3\cos\theta \cdot 3\cos\theta \, d\theta = 9 \int \cos^2\theta \, d\theta$$

$$= 9 \cdot \int \frac{1 + \cos(2\theta)}{2} \, d\theta = \frac{9}{2} \int \left(1 + \cos(2\theta)\right) d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

Double angle identity

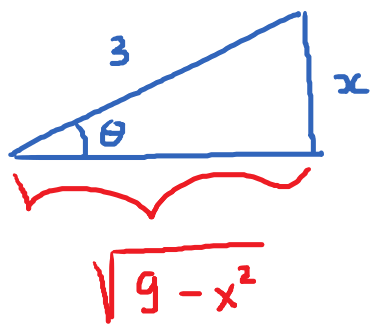
$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{9}{2}\theta + \frac{9\sin(2\theta)}{4} + C = \frac{9}{2}\theta + \frac{9\sin\theta\boxed{\cos\theta}}{2} + C$$

Q: How to convert this back to x?

$x = 3\sin\theta \, . \longrightarrow \boxed{\sin\theta = \frac{x}{3}}$

$\theta = \arcsin\left(\frac{x}{3}\right)$



$\sin\theta = \frac{x}{3}$

$\cos\theta = ?$

So, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

Ans: $\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$

$= \boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C}$

$\int\sqrt{9-x^2}\,dx$

Ex. Find $\int_0^{\sqrt{5}}\sqrt{5-x^2}\,dx$

Trig sub: $x = \sqrt{5}\sin\theta$.

$\int\sqrt{5}\cos\theta\cdot\sqrt{5}\cos\theta\,d\theta = 5\int\cos^2\theta\,d\theta$

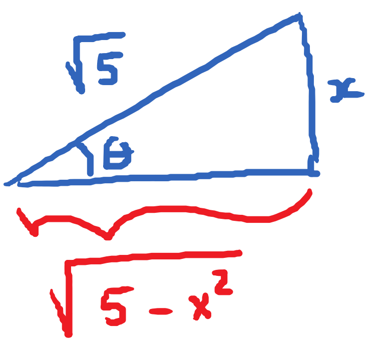
$\longrightarrow$ Integrate to $\frac{5}{2}\theta + \frac{5\sin(2\theta)}{4}$

$$= \boxed{\frac{5}{2}\theta + \frac{5\sin\theta\cos\theta}{2}}$$

$\longrightarrow$ Convert to x:

$$\frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) + \underline{\qquad\qquad}$$

$x = \sqrt{5}\cdot\sin\theta$; $\sin\theta = \frac{x}{\sqrt{5}}$



$$\cos\theta = \frac{\sqrt{5-x^2}}{\sqrt{5}}$$

$$\frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) + \frac{\cancel{5}}{2}\cdot\frac{x}{\cancel{\sqrt{5}}}\cdot\frac{\sqrt{5-x^2}}{\cancel{\sqrt{5}}}$$

$$\left(\frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) + \frac{1}{2}x\cdot\sqrt{5-x^2}\right)\Bigg|_0^{\sqrt{5}}$$