

3.4. Partial Fractions Decomposition

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1:02 PM

Goal: Apply this technique to find integrals of rational functions.

Integrals of the form $\int \frac{P(x)}{Q(x)} dx$

where P, Q are polynomials.

Key integrals: $\int \frac{1}{x} dx = \ln|x| + C$

$$\int \frac{1}{u} du = \ln|u| + C.$$

E.g. $\int \frac{1}{x+2018} dx = \ln|x+2018| + C.$

$$\int \frac{1}{x^n} dx; n \neq 1 \longrightarrow \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C$$

$$\int \frac{du}{u^n}; n \neq 1 \longrightarrow \frac{u^{-n+1}}{-n+1} + C.$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C.$$

$$\int \frac{du}{1+u^2} = \arctan(u) + C.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

Today, $\int \frac{P(x)}{Q(x)} dx$ in general.

1st thing to do when dealing with these integrals:

If $\text{degree } P(x) \geq \text{degree of } Q(x)$, then
we can do long division (or if $Q(x)$ has the
form $x \pm a$, synthetic division) to simplify
the expression.

E.g. $\int \frac{x^2 + 3x + 5}{x + 1} dx.$

degree of top = 2 > degree of bottom = 1

→ long division

$$\begin{array}{r}
 \boxed{x} + 2 \quad \text{quotient} \\
 \boxed{x} + 1 \overline{) \boxed{x^2} + 3x + 5} \\
 \underline{-(x^2 + x)} \\
 \boxed{2x} + 5 \\
 \underline{-(2x + 2)} \\
 \boxed{3} \quad \text{Remainder}
 \end{array}$$

$$\frac{x^2}{x} = x$$

$$\frac{2x}{x} = 2$$

$$\frac{x^2 + 3x + 5}{x + 1} = (x + 2) + \frac{3}{x + 1}$$

$$\int \frac{x^2 + 3x + 5}{x + 1} dx = \int \left(x + 2 + \frac{3}{x + 1} \right) dx$$

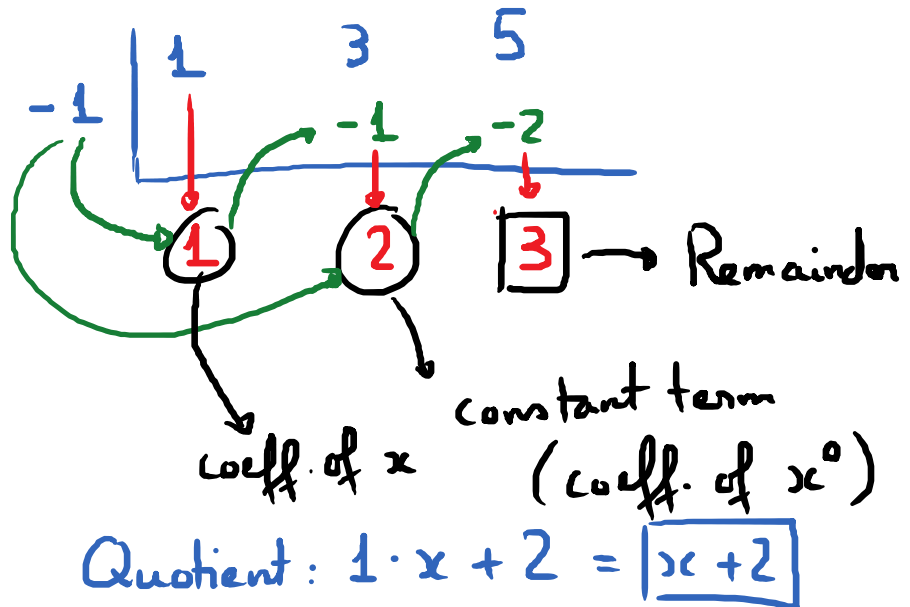
$$= \int (x+2) dx + \int \frac{3}{x+1} dx$$

$$= \boxed{\frac{x^2}{2} + 2x + 3 \ln|x+1| + C}$$

Shortcut to long division if bottom = $x \pm a$.

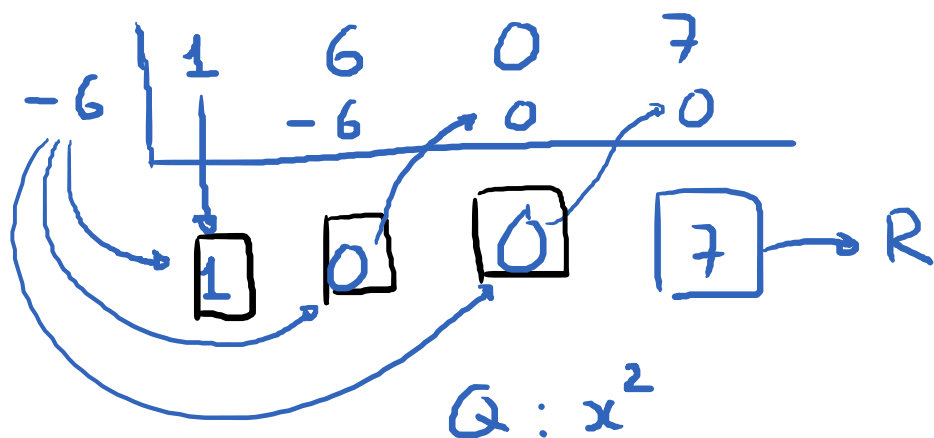
→ Synthetic Division.

$$\frac{x^2 + 3x + 5}{x + 1}$$



$$(x+2) + \frac{3}{x+1} \rightarrow \text{integrate.}$$

E.g. $\int \frac{x^3 + 6x^2 + 7}{x+6} dx$



$$\int \left(x^2 + \frac{7}{x+6} \right) dx = \boxed{\frac{x^3}{3} + 7 \ln|x+6| + C}$$

Now, assume $\deg P(x) < \deg Q(x)$.

Find $\int \frac{P(x)}{Q(x)} dx$

→ Factor $Q(x)$

After we finish factoring $Q(x)$ completely, there are a few different scenarios.

① $Q(x)$ can be factored into a product of distinct linear factors.

E.g. $\int \frac{3x+2}{x^3 - x^2 - 2x} dx$

$$= \int \frac{3x+2}{x(x^2 - x - 2)} dx$$

$$= \int \frac{3x+2}{\boxed{x} \boxed{(x-2)} \boxed{(x+1)}} dx$$

Completely factor bottom

→ product of 3 distinct linear factors

→ Partial Fractions Decomposition.

We want to decompose:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

Q: What should A , B , and C be in order for the equation above holds for all x ?

→ Multiply both sides by the common denominator $x(x-2)(x+1)$

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

We want to find A , B and C s.t. $LHS = RHS$ for all values of x .

→ Strategy: figure out "nice" values of x to plug in both sides until we get A, B, C .

Plug in $x = 2$: $8 = 6B \rightarrow B = \frac{4}{3}$

$$\text{Plug } x = 0 \text{ into both sides: } 2 = -2A \rightarrow \boxed{A = -1}$$

$$\text{Plug } x = -1 \text{ into both sides: } -1 = 3C \rightarrow \boxed{C = -\frac{1}{3}}$$

$$A = -1; B = \frac{4}{3}; C = -\frac{1}{3}.$$

→ Decomposition:

$$\int \frac{3x+2}{x(x-2)(x+1)} = \int \frac{-1}{x} + \int \frac{4/3}{x-2} + \int \frac{-1/3}{x+1}$$

$$= -1 \int \frac{dx}{x} + \frac{4}{3} \int \frac{dx}{x-2} - \frac{1}{3} \int \frac{dx}{x+1}$$

$$= \boxed{-\ln|x| + \frac{4}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C}$$