3.4. Partial Fractions Decomposition

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Goal: Apply this technique to find integrals of rational functions.

Integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$

where P, Q are polynomials.

Key integrals: $\int \frac{1}{x}\,dx = \ln|x| + C$

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

E.g. $\int \frac{1}{x+2018}\,dx = \ln|x+2018| + C.$

$$\int \frac{1}{x^n}\,dx \,;\; n \neq 1 \longrightarrow \int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1} + C$$

$$\int \frac{du}{u^n} \,;\; n \neq 1 \longrightarrow \frac{u^{-n+1}}{-n+1} + C.$$

$$\int \frac{dx}{1+x^2} = \arctan(x) + C.$$

$$\int \frac{du}{1+u^2} = \arctan(u) + C.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

Today, $\int \frac{P(x)}{Q(x)}\,dx$ in general.

1^{st} thing to do when dealing with these integrals:

If $\boxed{\text{degree } P(x) \geq \text{degree of } Q(x)}$, then we can do $\boxed{\text{long division}}$ (or if $Q(x)$ has the form $x \pm a$, synthetic division) to simplify the expression.

E.g. $\displaystyle\int \frac{x^2+3x+5}{x+1}\,dx.$

degree of top $= 2 >$ degree of bottom $= 1$

$\longrightarrow$ long division

quotient

$$\begin{array}{r|l} & \boxed{x+2} \\ \hline \boxed{x}+1 & \boxed{x^2}+3x+5 \\ & -(x^2+x) \\ \hline & \quad \boxed{2x}+5 \\ & \quad -(2x+2) \\ \hline & \qquad\quad \boxed{3} \longrightarrow \text{Remainder} \end{array}$$

$$\frac{x^2}{x} = x$$

$$\frac{2x}{x} = 2$$

$$\frac{x^2+3x+5}{x+1} = (x+2) + \frac{3}{x+1}$$

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int\left(x+2+\frac{3}{x+1}\right)dx$$

$$= \int (x+2)\,dx + \int \frac{3}{x+1}\,dx$$

$$= \boxed{\frac{x^2}{2} + 2x + 3\ln|x+1| + C}$$

Short-cut to long division if bottom $= x \pm a$.

$\rightarrow$ Synthetic Division.

$$\frac{x^2+3x+5}{x+1}.$$

$$\begin{array}{r|rrr} -1 & 1 & 3 & 5 \\ & & -1 & -2 \\ \hline & 1 & 2 & \boxed{3} \end{array}$$

$\boxed{3} \rightarrow$ Remainder

1: coeff. of x

2: constant term (coeff. of x^0)

Quotient: $1 \cdot x + 2 = \boxed{x+2}$

$$(x+2) + \frac{3}{x+1} \longrightarrow \text{integrate.}$$

E.g. $\displaystyle\int \frac{x^3+6x^2+7}{x+6}\,dx$

$$\begin{array}{r|rrrr} & 1 & 6 & 0 & 7 \\ -6 & & -6 & 0 & 0 \\ \hline & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{7} \rightarrow R \end{array}$$

$Q: x^2$

$$\int\left(x^2+\frac{7}{x+6}\right)dx = \boxed{\frac{x^3}{3}+7\ln|x+6|+C}$$

Now, assume $\deg P(x) < \deg Q(x)$.

Find $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$

$\longrightarrow$ $\boxed{\text{Factor } Q(x)}$

After we finish factoring $Q(x)$ completely, there are a few different scenarios.

① $Q(x)$ can be factored into a product of distinct linear factors.

E.g. $$\int \frac{3x+2}{x^3 - x^2 - 2x}\,dx$$

$$= \int \frac{3x+2}{x(x^2 - x - 2)}\,dx$$

$$= \int \frac{3x+2}{\boxed{x}\boxed{(x-2)}\boxed{(x+1)}}\,dx \longrightarrow \text{product of 3 distinct linear factors}$$

Completely factor bottom

$\longrightarrow$ Partial Fractions Decomposition.

We want to decompose:

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

Q: What should A, B, and C be in order for the equation above holds for all x?

→ Multiply both sides by the common denominator $x(x-2)(x+1)$

$$\boxed{3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}$$

We want to find A, B and C s.t. LHS = RHS for all values of x.

→ Strategy: figure out "nice" values of x to plug in both sides until we get A, B, C.

Plug in $x=2$: $8 = 6B \rightarrow \boxed{B = \frac{4}{3}}$

Plug $x=0$ into both sides: $2=-2A \rightarrow \boxed{A=-1}$

Plug $x=-1$ into both sides: $-1=3C \rightarrow \boxed{C=-\frac{1}{3}}$

$A=-1$; $B=\frac{4}{3}$; $C=-\frac{1}{3}$.

$\longrightarrow$ Decomposition:

$$\int \frac{3x+2}{x(x-2)(x+1)} = \int \frac{-1}{x} + \int \frac{4/3}{x-2} + \int \frac{-1/3}{x+1}$$

$$= -1\int \frac{dx}{x} + \frac{4}{3}\int \frac{dx}{x-2} - \frac{1}{3}\int \frac{dx}{x+1}$$

$$= \boxed{-\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C}$$