Thursday, February 22, 2018 1:53 PM

Another method to find A, B and C in the decomposition.

(Method of equating coefficients)

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2)$$

$$= A(x^2-x-2) + B(x^2+x) + C(x^2-2x)$$

$$= (A+B+C)x^2 + (-A+B-2C)x + (-2A)$$

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x + (-2A)$$

$2 = -2A$; $3 = -A+B-2C$; $0 = A+B+C$

(constant term) (coeff. for x) (coeff for x^2)

$\boxed{A=-1}$

$3 = 1 + B - 2C$ $\qquad$ $0 = -1 + B + C$

$B - 2C = 2$ $\qquad$ $B + C = 1$

$1 - C - 2C = 2$ $\qquad$ $\boxed{B = 4/3}$ $\qquad$ $B = 1 - C$

$-3C = 1 \rightarrow \boxed{C = -1/3}$

Ex. HW #13 $\displaystyle\int \frac{\sin(x)}{\cos^2(x)+\cos(x)-30}\,dx.$

Solved in class.

Summary:

Case 1 of Partial Fractions Decomposition:
Distinct (non-repeated) linear factors.

$$\int \frac{P(x)}{Q(x)}\,dx \; ; \; \deg P < \deg Q$$

$Q(x)$ can be factored into non-repeated linear factors.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

Strategy: Find numbers $A_1, A_2, \ldots, A_n$ such that:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{ax_n+b_n}$$

$\rightarrow$ using Strategic substitution or Equating coefficients to find $A_1, \ldots, A_n$.

Then

$$\int \frac{P(x)}{Q(x)}\,dx = A_1 \int \frac{dx}{a_1 x + b_1} + \cdots + A_n \int \frac{dx}{a_n x + b_n}$$

Case 2 of P.F.D.

The factorization of $Q(x)$ has repeated linear factors.

$$\int \frac{4x^2}{(x-1)(x-2)^2}$$

Repeated linear factor. $(x-2)$ is repeated.

$\longrightarrow$ The decomposition is different.

$$\frac{4x^2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Multiply by $(x-1)(x-2)^2$:

$$4x^2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$$

$x=1$: $\boxed{4 = A}$

$x=2$: $\boxed{16 = C}$

$x=0$:

$$0 = 4A + 2B - C$$

$$0 = 16 + 2B - 16$$

$\rightarrow \boxed{B = 0}$

$\longrightarrow$ Decomposition:

$$\int \frac{4x^2}{(x-1)(x-2)^2} = \int \frac{4}{x-1} + \int \frac{16}{(x-2)^2}$$

$$= 4 \int \frac{dx}{x-1} + 16 \cdot \int \frac{dx}{(x-2)^2}$$

$$= 4\int \frac{dx}{x-1} + 16 \cdot \int (x-2)^{-2}\,dx$$

$$= 4\ln|x-1| + 16 \cdot \frac{(x-2)^{-1}}{-1} + C$$

$$= 4\ln|x-1| - \frac{16}{x-2} + C$$

Summary: If $Q(x)$ has a repeated linear factor, say $(ax+b)^m$; $(ax+b)$ appears m times, then the decomposition for this repeated factor is:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

Case 3: $Q(x)$ has an irreducible quadratic factor (a quadratic factor that cannot be factored over the real #)

E.g. $\displaystyle\int \frac{dx}{x^2+4x+18}$

$\longrightarrow$ Complete the square :

$$x^2+4x+18 = \underbrace{x^2+4x+4}_{(x+2)^2}+14$$
$$= (x+2)^2+14$$

$$\int \frac{dx}{(x+2)^2+14}.$$ let $u = x+2.$

$du = dx$

$$\int \frac{du}{14+u^2} = \frac{1}{\sqrt{14}}\cdot \arctan\left(\frac{u}{\sqrt{14}}\right)+C$$

$$= \frac{1}{\sqrt{14}}\cdot \arctan\left(\frac{x+2}{\sqrt{14}}\right)+C$$

Sub case: we have a product of distinct irreducible quadratic factors in $Q(x)$

E.g. $\displaystyle\int \frac{x}{(x^2+1)(x^2+2)}$

Decomposition:

$$\frac{x}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

Find A, B, C, D.

Multiply both sides by $(x^2+1)(x^2+2)$

$$x = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$x = Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D$$

$$x = \boxed{(A+C)}x^3 + \boxed{(B+D)}x^2 + \boxed{(2A+C)}x + (2B+D)$$

$$\left.\begin{array}{l} A+C=0 \ ; \ B+D=0 \\ 2A+C=1 \ ; \ 2B+D=0 \end{array}\right\} \begin{array}{l} A=1 \ ; \ C=-1 \\ B=D=0 \end{array}$$

$$\int \frac{x}{(x^2+1)(x^2+2)} = \int \frac{x}{x^2+1} - \int \frac{x}{x^2+2}$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+2}\,dx .$$

$u = x^2+1$; $du = 2x\,dx$ $w = x^2+2$
$dw = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{dw}{w}$$

$$= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|w|$$

$$= \boxed{\frac{1}{2}\ln|x^2+1| - \frac{1}{2}\ln|x^2+2| + C}$$

Summary: The decomposition of an irreducible quadratic factor ax^2+bx+c has the form:

$$\frac{Ax+B}{ax^2+bx+c}.$$