3.6 Numerical Integration

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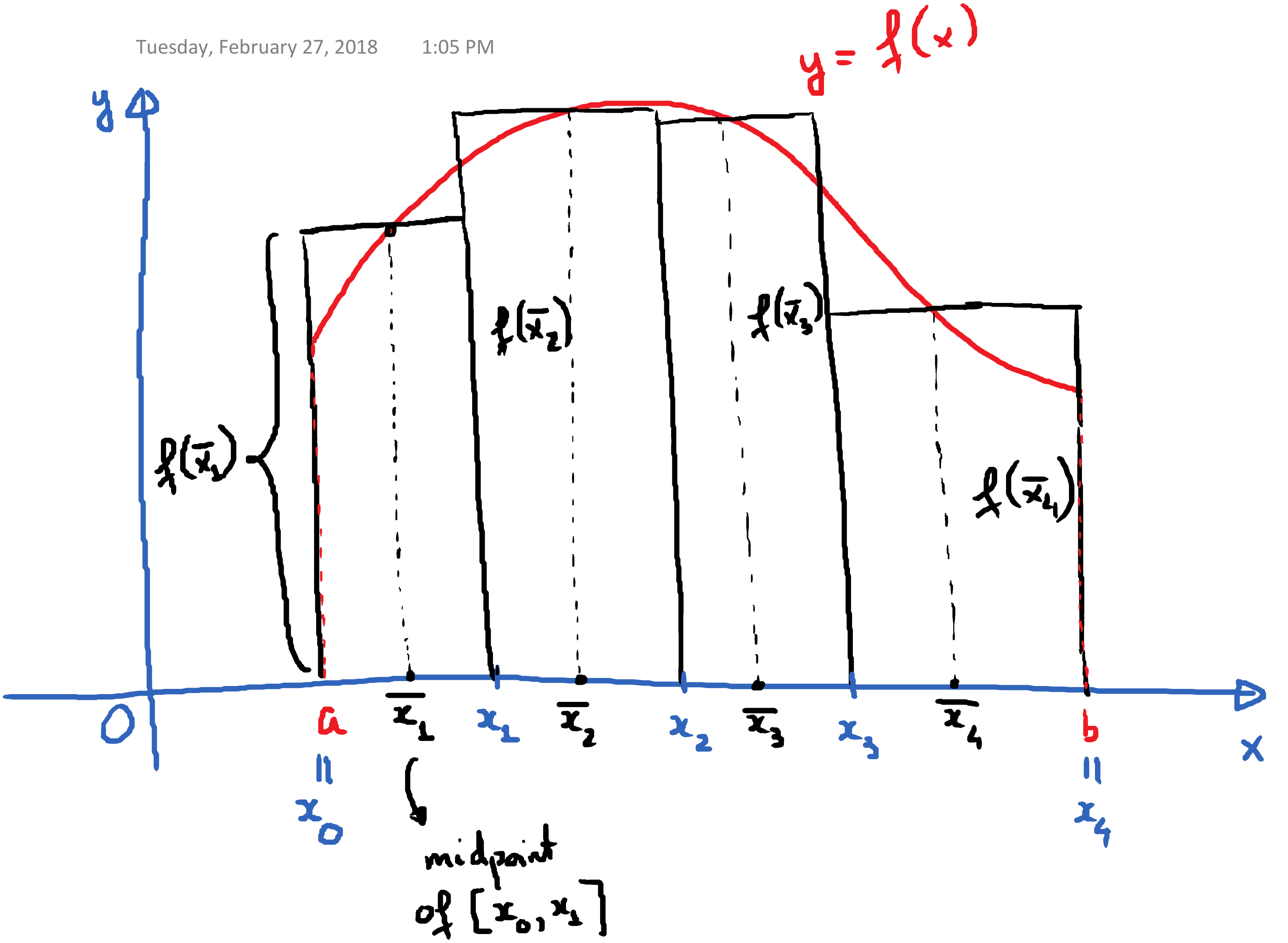
Goals:

1. Midpoint Rule
2. Trapezoid Rule
3. Simpson's Rule
4. Error estimates for these rules.

Midpoint Rule:

$$\int_a^b f(x)\,dx, \quad f(x) \geq 0.$$



$$\int_a^b f(x)\,dx = \text{area under } y = f(x) \text{ from } x = a \text{ to } x = b$$

$$\simeq \text{Sum of areas of rectangles.}$$

In this case, we will denote the sum of the areas of the 4 rectangles in the picture by M_4 → midpoint rule, 4 → 4 rectangles.

Width of each rectangle for $M_4 = \dfrac{b-a}{4} = \Delta x$.

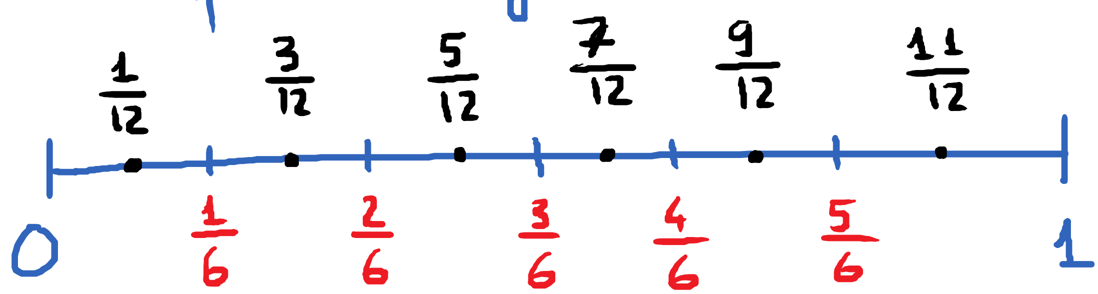
The height of rectangles are $f(\overline{x_1}), f(\overline{x_2}), f(\overline{x_3}), f(\overline{x_4})$.

Hence, $M_4 = f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + f(\overline{x_3})\Delta x + f(\overline{x_4})\Delta x$

$$\boxed{M_4 = \left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + f(\overline{x_4})\right]\cdot\Delta x}$$

E.g. $\int_0^1 \ln(x+5)\,dx$.

Use the midpoint rule with $n=6$ to find an estimate for this integral.



$M_6 = \left[f\left(\frac{1}{12}\right) + f\left(\frac{1}{4}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{11}{12}\right)\right]\cdot(\Delta x)$

$\boxed{\Delta x = \frac{1}{6}}$

$\approx \boxed{1.7034}$

Process and Formula for the midpoint rule

$\int_a^b f(x)\,dx$, $f(x) \geq 0$. Use n rectangles.

Step 1: Subdivide $[a,b]$ into n subintervals of equal length. The length of each subinterval is:

$$\Delta x = \frac{b-a}{n}$$

Step 2: Find the sum:

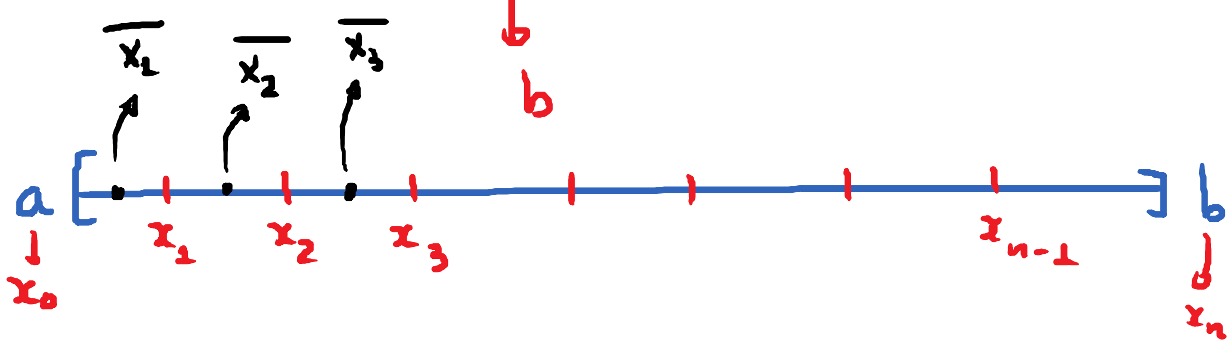
$$\boxed{M_n = \left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right] \cdot \Delta x}$$

M_n is the approximation to the integral using the midpoint rule with n rectangles.

Here, $\bar{x}_1, \ldots, \bar{x}_n$ are the midpoints of the interval

$[x_0, x_1], \ldots, [x_{n-1}, x_n]$

$x_0 = a$, $x_n = b$



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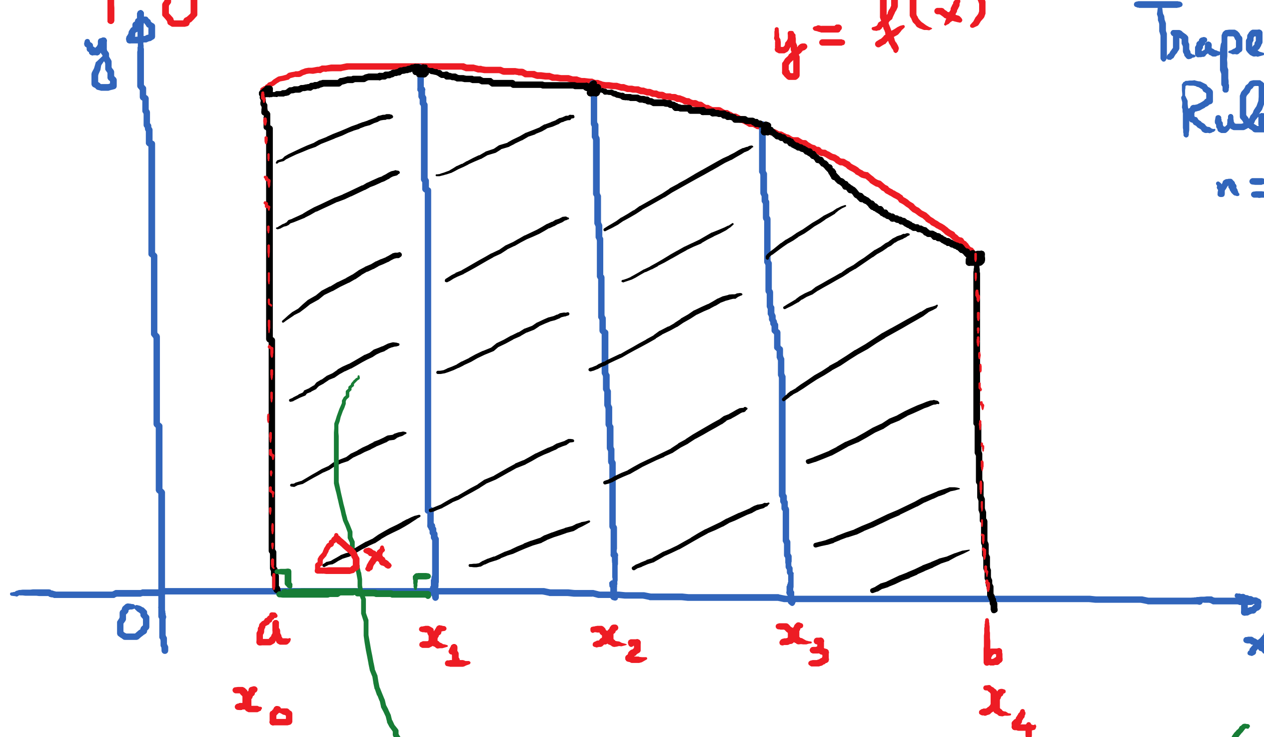
Trapezoid Rule



Trapezoid Rule with $n=4$

Area of this trapezoid $= \frac{1}{2}\Delta x\left(f(a)+f(x_1)\right)$

$= \frac{\Delta x}{2}\left(f(x_0)+f(x_1)\right)$

using trapezoid rule

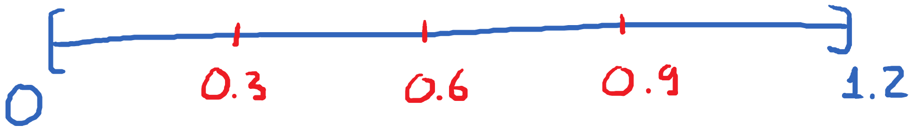
Sum of the areas of the 4 trapezoids:

$$T_4 = \frac{\Delta x}{2}\left[f(x_0)+f(x_1)+f(x_1)+f(x_2)+f(x_2)+f(x_3) + f(x_3)+f(x_4)\right]$$

n

$$\boxed{T_4 = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]}$$

E.g. $\int_0^{1.2} \sin(x^2)\,dx$. Use trapezoid rule with $n=4$ to estimate this integral.



$$\Delta x = \frac{1.2-0}{4} = 0.3$$

$$T_4 = \frac{0.3}{2}\left[\sin(0^2) + 2\sin((0.3)^2) + 2\sin((0.6)^2) + 2\sin((0.9)^2) + \sin((1.2)^2)\right]$$

$$\approx 0.49865$$

In general, the formula for the Trapezoid rule with n trapezoids is

$$\boxed{T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]}$$

$$\boxed{\Delta x = \frac{b-a}{n}}$$

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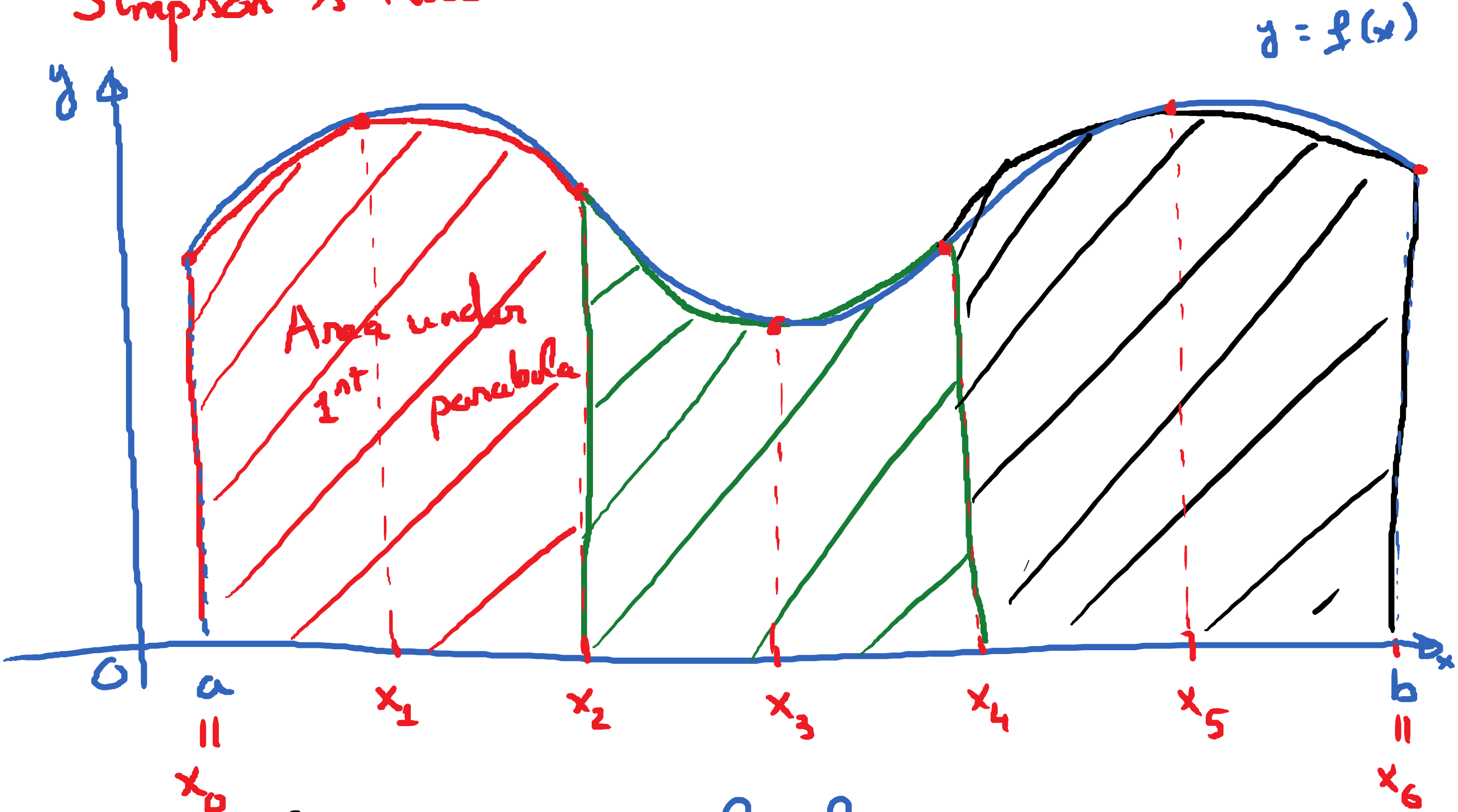
Simpson's Rule



Simpson Rule

Simpson's rule for n = 6.

$$\Delta x = \frac{b-a}{6}$$

S_6 = Sum of areas under 3 parabolas

n

$$= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$