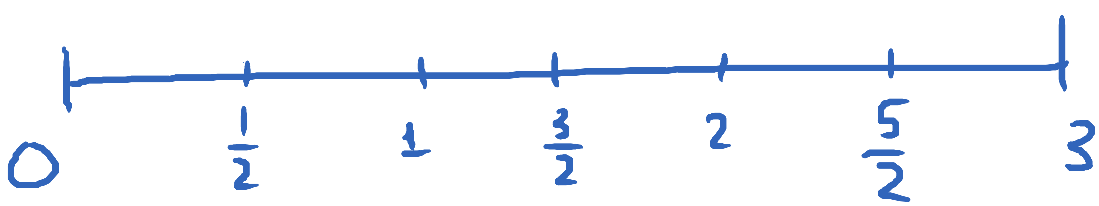
In general, the formula for Simpson's rule with n subintervals (n is always even) is:

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

E.g. Use Simpson's rule rule with $n=6$ to estimate

$$\int_0^3 \frac{dx}{1+x^3}.$$

$$\Delta x = \frac{3}{6} = \frac{1}{2}.$$



$$S_6 = \frac{\frac{1}{2}}{3}\Big[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\Big]$$

$$\approx 1.1614$$

Error Estimates for these numerical integration methods

Estimate $\int_a^b f(x)\,dx$

using the Midpoint Rule with n subintervals : M_n

______ Trapezoid Rule ______ : T_n

______ Simpson's Rule ______ : S_n

Exact errors for each of these methods:

Exact error for Midpoint:

$$E_M = \underbrace{\int_a^b f(x)\,dx}_{\text{actual value of integral}} - \underbrace{M_n}_{\text{estimate to integral}}$$

Exact error for Trapezoid : $E_T = \int_a^b f(x)\,dx - T_n$

______ Simpson : $E_S = \int_a^b f(x)\,dx - S_n$.

→ We normally cannot find these exact errors in practice

→ Find upper bounds for these errors.

Error Bounds for E_M, E_T and E_S.

We have:

$$|E_M| \leq \frac{K\cdot(b-a)^3}{24n^2}$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

where K is a number such that $|f''(x)| \leq K$ for all x in $[a,b]$; i.e., K is an upper bound for $|f''(x)|$ on $[a,b]$

For Simpson's rule: $|E_S| \leq \dfrac{K(b-a)^5}{180n^4}$.

where K is a number such that

$$|f^{(4)}(x)| \leq K \text{ for all } x \text{ in } [a,b].$$

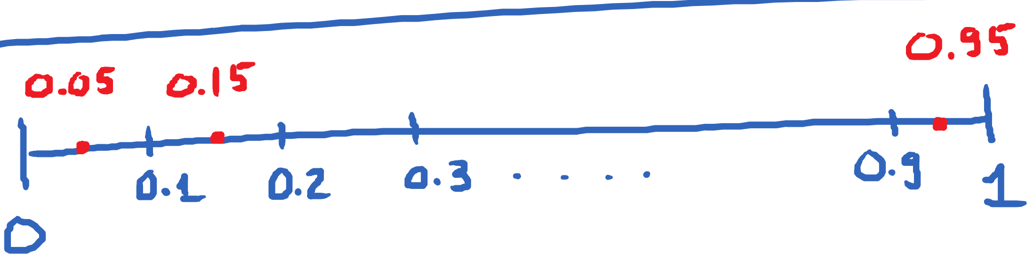
E.g. $\int_0^1 e^{x^2}\,dx$.

(1) Use the midpoint rule for $n=10$ to estimate this integral. $\longrightarrow M_{10}$.

(2) Find an upper bound for the error in using M_{10} to estimate the integral, i.e., find an upper bound for $E_{M_{10}}$.

---

(1)



$$\Delta x = \frac{1-0}{10} = \frac{1}{10} = 0.1$$

$$M_{10} = (0.1)\cdot\left[f(0.05) + f(0.15) + \cdots + f(0.95)\right]$$

$$M_{10} \approx 1.460393.$$

Tuesday, February 27, 2018 2:29 PM

② Find an upper bound for $|E_{M_{10}}|$

$$\left(|E_{M_{10}}| = \left| \int_0^1 e^{x^2}\,dx - M_{10} \right| \right)$$

By the error bound formula:

$$|E_{M_{10}}| \le \frac{K \cdot (b - a)^3}{24 n^2}$$

1, 0, 10

$b = 1, \quad a = 0; \quad n = 10.$

Need K? K is an upper bound for $|f''(x)|$ on $[0,1]$. $f(x) = e^{x^2}$.

$f'(x) = 2x e^{x^2}$. $\quad \left(\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx} \right)$

$f''(x) = 2\left[1 \cdot \boxed{e^{x^2}} + x \cdot 2x \cdot \boxed{e^{x^2}}\right]$

$f''(x) = 2e^{x^2}(1 + 2x^2)$ → this is already positive, no need for abs.

Q: What is the maximum value of $f''(x)$ on $[0,1]$?

→ find K.

Since $f''(x) = 2e^{x^2}(1+2x^2)$ is an increasing function on $[0,1]$, the largest value of the function is at $x = 1$. So, $K = |f''(1)| = |2e^1 \cdot (1+2)|$
$= |6e| = 6e.$

So, $|E_{M_{10}}| \leq \dfrac{6e \cdot (1-0)^3}{24 \cdot (10)^2} \approx \boxed{0.0067}$

this tells you that the error of the approximation in 1 cannot be more than this number.

E.g. Using the error bound for the Simpson's Rule.
Determine the smallest value of n such that the Simpson approximation to $\int_1^2 \frac{1}{x}\,dx$ is accurate to within 0.0001?

$$|E_S| \le \boxed{\frac{K(b-a)^5}{180 n^4}}$$

<u>Want:</u> find n so that $\dfrac{\underbrace{K}_{?}(\overbrace{b}^{2}-\overbrace{a}^{1})^5}{180 n^4} \le 0.0001$

K is an upper bound for $|f^{(4)}(x)|$ on $[1,2]$.

$f(x) = \dfrac{1}{x}$; $f'(x) = -x^{-2}$; $f''(x) = 2x^{-3}$

$f^{(3)}(x) = -6x^{-4}$; $f^{(4)}(x) = 24x^{-5} = \dfrac{24}{x^5}$.

Since $f^{(4)}(x) = \dfrac{24}{x^5}$ is a decreasing function on $[1,2]$, the maximum value of it on $[1,2]$ is : $\dfrac{24}{(1)^5} = 24$.

Choose $K = 24$.

So, $\dfrac{24}{180 n^4} \le 0.0001 \longrightarrow$ solve this inequality for n.

$$24 \le (0.0001) \cdot 180 n^4$$

$$\frac{24}{0.018} \leq n^4 \; ; \; \sqrt[4]{\frac{24}{0.018}} \leq n$$

$$6.042 \leq n.$$

$$\longrightarrow \boxed{n = 7}$$