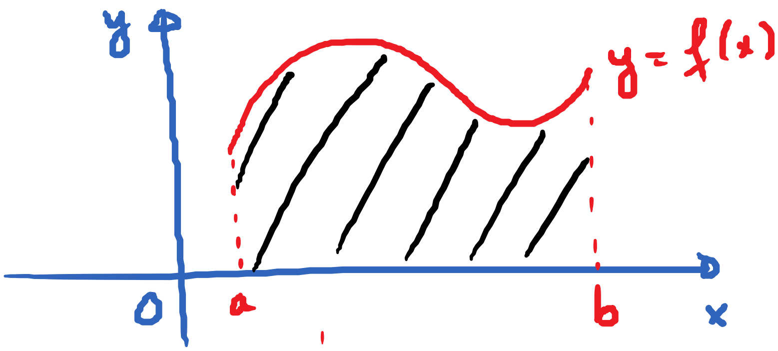
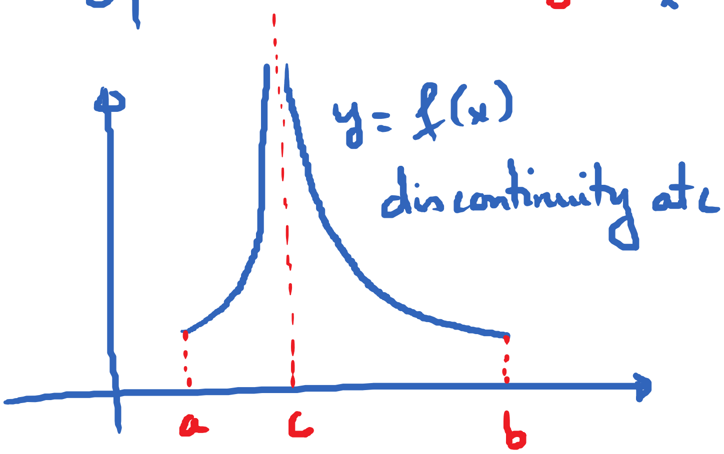
3.7. Improper Integrals.

Thursday, March 1, 2018 1:00 PM

$$\int_a^b f(x)\,dx \; : \quad \left.\begin{array}{l} a, b \text{ are finite \#'s} \\ f \text{ is continuous on } [a,b] \end{array}\right\} \text{proper integrals.}$$

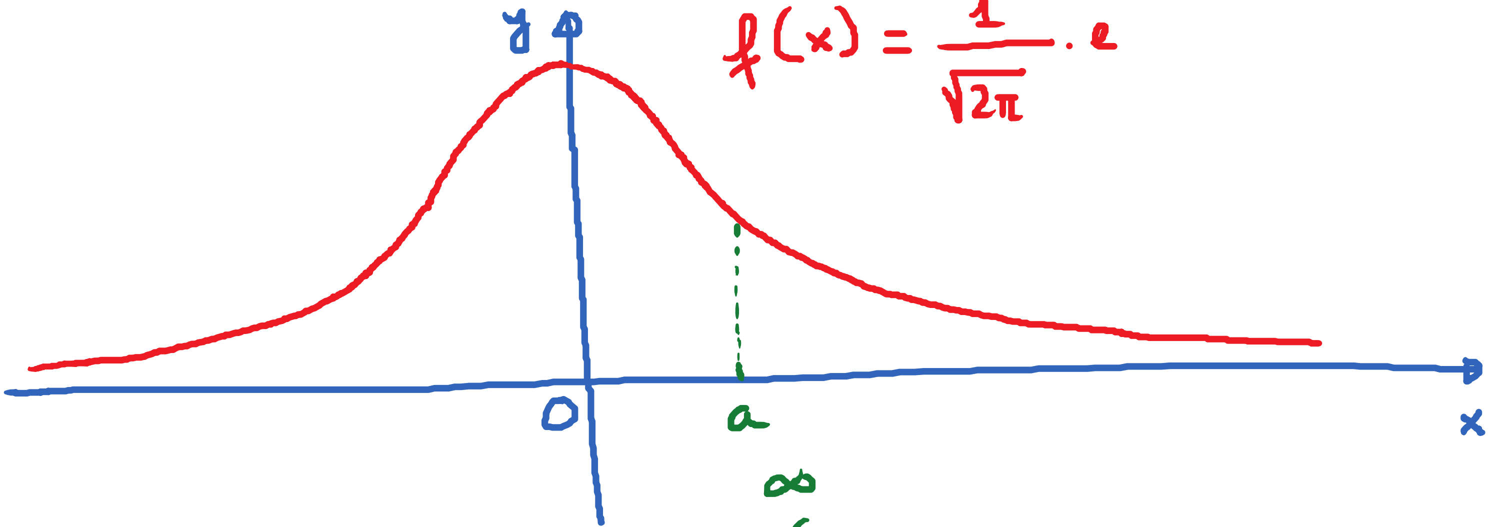




* If $a = \pm\infty$ or $b = \pm\infty$ or both ; or if f is not continuous on $[a,b]$, then we have an improper integrals.

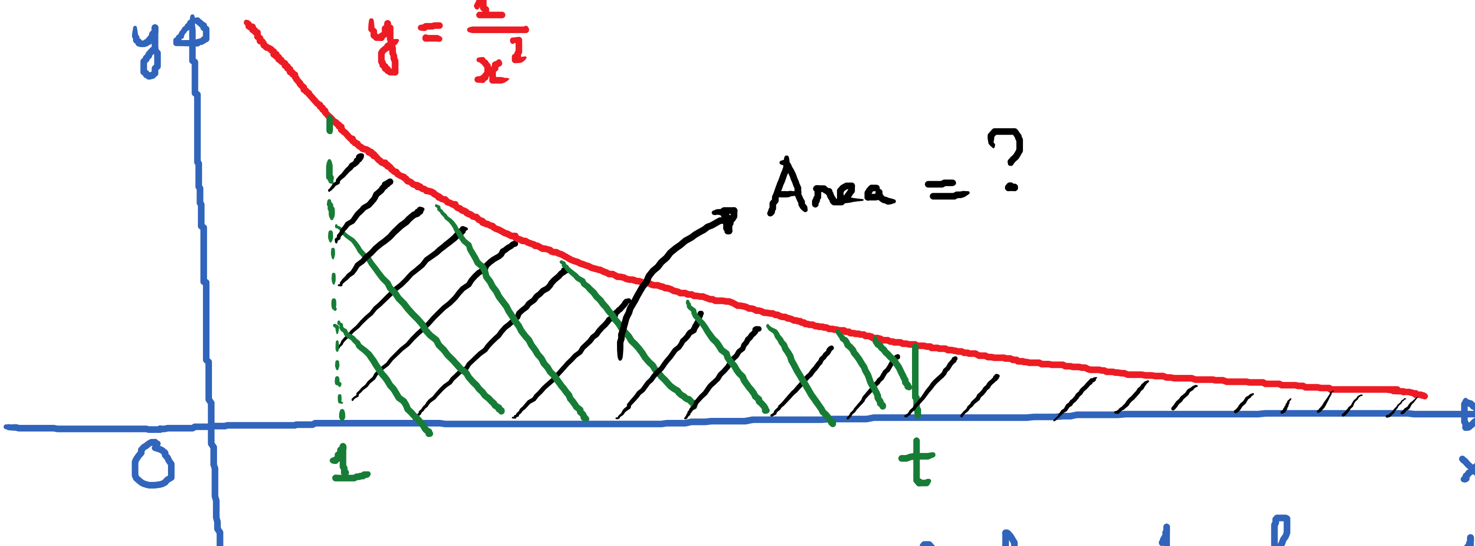
Thursday, March 1, 2018 1:36 PM

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$



$$\text{Probability}\,[X > a] = \int_a^{\infty} f(x)\,dx$$ ← improper integrals.

An example of a simple improper integral.



Find the area under the graph of $y = \frac{1}{x^2}$ from $x = 1$ to $x = t$:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^t$$

area under graph from 1 to t

$$= \left.\left(-\frac{1}{x}\right)\right|_1^t = \left(-\frac{1}{t}\right) - \left(-\frac{1}{1}\right)$$

$$\boxed{A(t)} = -\frac{1}{t} + 1$$

→ Area under the graph of $y = \frac{1}{x^2}$ from 1 to ∞ is the limit as t approaches infinity of the above expression:

$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(-\frac{1}{t} + 1\right) = \boxed{1}$$

0

We just calculated an improper integral:

$$\boxed{\int_1^\infty \frac{1}{x^2}\,dx = 1}$$

Type 1 Improper Integrals:

Integrals of the form: $\int_a^{\infty} f(x)\,dx$; $\int_{-\infty}^{b} f(x)\,dx$;

$\int_{-\infty}^{\infty} f(x)\,dx \longrightarrow$ Infinite Integrals.

So, by definition:

$$\boxed{\int_a^{\infty} f(x)\,dx := \lim_{t\to\infty} \int_a^{t} f(x)\,dx}$$

provided that the limit exists as a finite #.

If it does not exist, then we say that $\int_a^{\infty} f(x)\,dx$ diverges.

Similarly,

$$\boxed{\int_{-\infty}^{b} f(x)\,dx := \lim_{t\to-\infty} \int_{t}^{b} f(x)\,dx}$$

provided that the limit exists as a finite #

Finally,

$$\int_{-\infty}^{\infty} f(x)\,dx = \boxed{\int_{-\infty}^{a} f(x)\,dx} + \boxed{\int_{a}^{\infty} f(x)\,dx}$$

where a is any real #

If both integrals exist and finite, then $\int_{-\infty}^{\infty} f(x)\,dx$ exists.

Otherwise, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Ex. Determine whether the improper integral diverges or converges. If it converges, find the integral.

(1) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ (2) $\displaystyle\int_{-\infty}^{0} x e^{x}\,dx.$

(Hint: L'Hopital Rule to find limit)

(3) $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{0} + \int_{0}^{\infty}$

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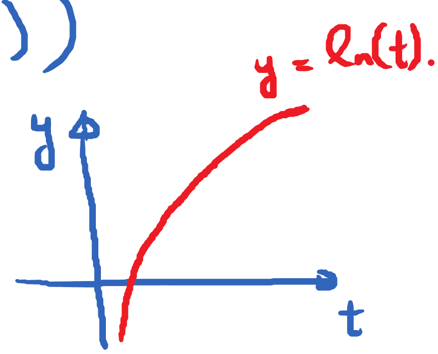
(1) $$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \boxed{\int_1^{t} \frac{1}{x}\,dx} = \lim_{t\to\infty} \Big[\ln(x)\Big]\Big|_1^t$$

$$= \lim_{t\to\infty} \big(\ln(t) - \ln(1)\big)$$

$$= \lim_{t\to\infty} \ln(t) = \infty$$



So, $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges.